The Sine-Gordon model in hyperbolic space and the AdS/CFT correspondence

Master Thesis

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Universität Leipzig

Erik Löffelholz (Matr. No. 3742085)

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Supervisor: Prof. Dr. Stefan Hollands
Second Examiner: Dr. Markus Fröb

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Chapter 1

Introduction

The Sine-Gordon model, a fundamental example within the field of integrable quantum field theories, offers a rich landscape for studying non-perturbative effects and operator correspondences within the AdS/CFT framework. The AdS/CFT correspondence, originally proposed by Maldacena, conjectures a remarkable connection between a theory of quantum gravity in $d+1$-dimensional Anti-de Sitter space and a d-dimensional conformal field theory CFT on the boundary of that space. This duality is essential for probing deep questions in quantum gravity, holography, and renormalization group flows [42], [38], [54].

In this thesis, one explores the AdS/CFT correspondence in the context of a two-dimensional Euclidean background geometry. The central focus is the study of perturbative effects in this framework. Perturbation theory in AdS is considered to be convenient for studying weakly coupled QFT dynamics beyond flat space, hence it is relevant for constructing corrections [57]. The sine-Gordon model is a well-known integrable field theory with applications in various areas of physics. Here, it is considered as a bulk theory in the AdS geometry, with its vertex operators representing key observables. In the framework, the primary motivation lies in understanding how bulk field interactions, particularly in models such as the Sine-Gordon theory, relate to operators in the boundary CFT, revealing structures such as logarithmic multiplets that enrich the operator spectrum.

To carry out perturbation theory in this setting, one utilizes the heat kernel method, which decomposes the relevant propagators into simpler components. This decomposition allows one to formulate a Gaussian measure, which forms the backbone of our perturbative expansion. Specifically, the heat kernel provides a systematic way to define Green's functions and regularize ultraviolet divergences, facilitating the computation of renormalized quantities.

The thesis is structured as follows. In Chapter 2, one lays the groundwork by introducing the geometry of a two-dimensional Euclidean Anti-de Sitter space using Poincaré coordinates, giving a concise overview of conformal field theories and logarithmic conformal field theories, concluding with a brief but essential introduction to the AdS/CFT correspondence. In Chapter 3, one discusses the behavior of scalar fields near the boundary, followed by a detailed construction of various propagators with a focus on their correspondence to a free CFT. The chapter concludes with the formulation of a spectral representation, setting the stage for the heat kernel regularization upon the bulk propagator. In Chapter 4, one applies the regularization scheme for a detailed perturbative treatment of the Sine-Gordon model in $\mathbb{E}\mathrm{AdS}^2$ space, computing the boundary expectation values of vertex operators with a focus on renormalizing to extract physical observables.

1.1 Notations and Conventions

We will work in natural units, where

$$\hbar = c = 1. \tag{1.1}$$

Thus in this system,

$$[\text{length}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}. \tag{1.2}$$

We use the metric tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1), \tag{1.3}$$

with Greek indices running over $0, 1, \ldots$ and roman indices denote only $1, 2, \ldots$. Repeated indices are summed in all cases. Vectors are denoted by

$$\vec{x} = x^\mu = (x^0, x^a). \tag{1.4}$$

Chapter 2

Preliminaries

To reliably work and discuss the AdS/CFT correspondence, it is necessary to introduce the considered types of physical models such as the Anti-de Sitter space model in the first section and Conformal Field Theories in the second section. For them, we specifically look at their topological structure and isometries, which should give a hint on the correspondence we define as a conclusion of this chapter,i.e. we are going to relate them to each other in order to understand the framework of the AdS/CFT correspondence. This construction will lay the groundwork in order to apply it to a quantum field theory.

2.1 Background geometry

First of all, we will discuss the background geometry which sets the stage of all future calculations in the thesis. We demand from our space or manifold to be isotropic and homogeneous, i.e. satisfying the Copernican principle. Therefore, the space looks the same everywhere and the metric does not change throughout the manifold. A more detailed introduction can be found in [16]. Thus we start by reviewing the model of a $d+1$-dimensional Euclidean Anti-de Sitter space ($\mathbb{E}\mathrm{AdS}^{d+1}$). In general, Anti-de Sitter spaces are realized by constant negative curvature and arise as a maximally symmetric solution to the vacuum Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{2.1}$$

where $G_{\mu\nu}$ denotes the Einstein tensor, $g_{\mu\nu}$ the metric tensor and Λ is the cosmological constant [21]. Spaces which arise by such particular exact solutions admit at most $\frac{(d+1)(d+2)}{2}$ linearly independent global Killing vector fields. Thus the Riemann curvature tensor for a maximally symmetric $d+1$-dimensional manifold is simplified to

$$R_{\mu\nu\rho\sigma} = \kappa \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right), \tag{2.2}$$

where we identify κ to be the normalized measure of the Ricci curvature

$$\kappa = \frac{R}{d(d+1)}, \tag{2.3}$$

and R is the Ricci scalar, fixed to be constant over the entire manifold. In order to describe physical objects living in this space, the line element provides a complete prescription of the underlying geometry, generically reading

$$\mathrm{d}s^2_{\mathbb{M}} = \eta_{AB}\,\mathrm{d}X^A\,\mathrm{d}X^B, \tag{2.4}$$

where η_{AB} is the metric tensor of some Minkowski space. By definition, metrics of maximally symmetric manifolds can always be put into the canonical form of (2.4).

In the case of a Euclidean AdS space, i.e. $\kappa < 0$, the line element (2.4) can be realized as one out of two[1] disconnected d-dimensional hyperbolic spaces $\mathbb{H}^d$, embedded into a $d+1$-dimensional ambient Minkowski space $\left(\mathbb{M}^{1,d} \simeq \mathbb{S}^1 \times \mathbb{R}^d\right)$ parametrized by the standard Cartesian coordinates X^A where $A = 0, 1, \ldots, d+1$, satisfying the constraint equation [35]

$$\eta_{AB}X^AX^B \equiv -\left(X^0\right)^2 + \sum_{i=1}^{d}\left(X^i\right)^2 = -L^2_{\mathrm{AdS}}, \tag{2.5}$$

with $L_{\mathrm{AdS}} \in \mathbb{R}$ denoting the AdS radius. Thus the line element of $\mathbb{M}^{1,d}$ naturally induces a metric tensor g_{ab} on the hyperboloid [32]

$$\mathrm{d}s^2_{\mathbb{H}^d} = -\left(\mathrm{d}X^0\right)^2 + \sum_{i=1}^{d}\left(\mathrm{d}X^i\right)^2. \tag{2.6}$$

By construction, the space has isometry $SO(1, d+1)$, and it is isotropic and homogeneous. The main task, important to the thesis is to locate the boundary of this space. A common procedure is to introduce an appropriate coordinate parametrization in which the Anti-de Sitter space can be described in a more efficient way in terms of its topological shape. Hence, we will be discussing first the most suitable parametrization to work in and afterwards roll back to the embedding space to discuss its formalism towards the boundary and even provide a connection to the parametrization.

2.1.1 Poincaré parametrization

It is possible to solve the constraint equation (2.5) in several systems of coordinates such as in global coordinates but AdS spaces do not possess a boundary in general since they are theoretically open spaces. However if a coordinate patch is considered and a boundary is assigned within its framework as they do not cover the whole space, one is able to provide a notion of a boundary formalism. For particular importance we will therefore consider the Poincaré patch. It provides various advantages as it makes the Poincaré subgroup of the conformal group manifest since the metric displays explicit Poincaré invariance in its transverse coordinates which is discussed in the next section. Another key feature is covered by its scale invariance. Thus, as it is of physical importance we will be considering Poincaré coordinates throughout the work.

[1] Only the $X^0 > 0$ hyperboloid is considered to be relevant.

The *Poincaré fundamental domain* in $d+1$ dimensions (PAdS^{d+1}) is introduced with the coordinate parametrization [21], [35]

$$\begin{aligned} X^0 &= \frac{1}{2z}\left(L_{\mathrm{AdS}}^2 + z^2 + \delta_{ab}x^a x^b\right), \quad a = 1, \ldots, d, \\ X^a &= \frac{L}{z}x^a, \\ X^{d+1} &= \frac{1}{2z}\left(L_{\mathrm{AdS}}^2 - z^2 - \delta_{ab}x^a x^b\right), \end{aligned} \tag{2.7}$$

for $\vec{x} = \{z, x^a\}$. Here $x^a \in \mathbb{R}^d$ are the d-dimensional spatial coordinates and $z \in (0, \infty)$ is the radial coordinate. The latter is a consequence of the embedding constraint (2.5) to keep the correct sign of X^0. Thus its induced metric is constructed on a subset $\mathbb{R}^d \times (0, \infty)$ of $\mathbb{R}^{d+1}$ in a Euclidean context [24],

$$\mathrm{d}s_{\mathbb{H}^d}^2 = \frac{L_{\mathrm{AdS}}^2}{z^2}\left(\mathrm{d}z^2 + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b\right). \tag{2.8}$$

Hence PAdS^{d+1} yields integration measure

$$\mathrm{d}X = \left(\frac{L_{\mathrm{AdS}}}{z}\right)^{d+1} \mathrm{d}z\,\mathrm{d}^d x. \tag{2.9}$$

Also the Poincaré coordinate patch immediately gives rise to the identity [26]

$$X^0 + X^2 = \frac{L_{\mathrm{AdS}}^2}{z} = \frac{L_{\mathrm{AdS}}X^a}{x^a} \tag{2.10}$$

displaying a constraint surface which splits the embedded hyperboloid into two disconnected hypersurfaces corresponding to half spaces, each of them assigned with the sign of z. As z has been defined to take only positive values, the upper-half space will be considered throughout the thesis as it is of most physical relevance in general.
By analyzing the constraint surface (2.10) and the coordinates of the Poincaré fundamental domain, it can be seen that $z = 0$ does not belong to the AdS space. Even more the induced metric (2.8) happens to degenerate at this point. Due to this fact, one follows the exclusion of the endpoint $z = 0$.

Nonetheless, a geometrical meaning can be assigned to $z = 0$ as there is no singularity in the curvature tensor [32] described in (2.2). Therefore it is considered to be the boundary of AdS space in Poincaré coordinates. Looking back at the induced metric (2.8), we even conclude its form to be conformal to a portion of Euclidean space, hence giving rise to a conformal factor $\Omega = \frac{z}{L_{\mathrm{AdS}}}$. Thus it appears to be conformally flat and even equivalent to an upper-half plane Euclidean metric as the limit $z \to 0$ is taken.
In physical terms, AdS space is of negative curvature, thus it is expected that the expansion of the space itself is decelerating, hence displaying the effect of a conformal boundary. While compactifying $\mathbb{R}$, it is sufficient enough to add a point at infinity where the space then can be conformally compactified into Euclidean space. Thus Euclidean AdS contains the conformal Euclidean space at infinity.
Even further, $\mathbb{H}^{d+1}$ is conformally equivalent to a $d+1$-dimensional disk, since

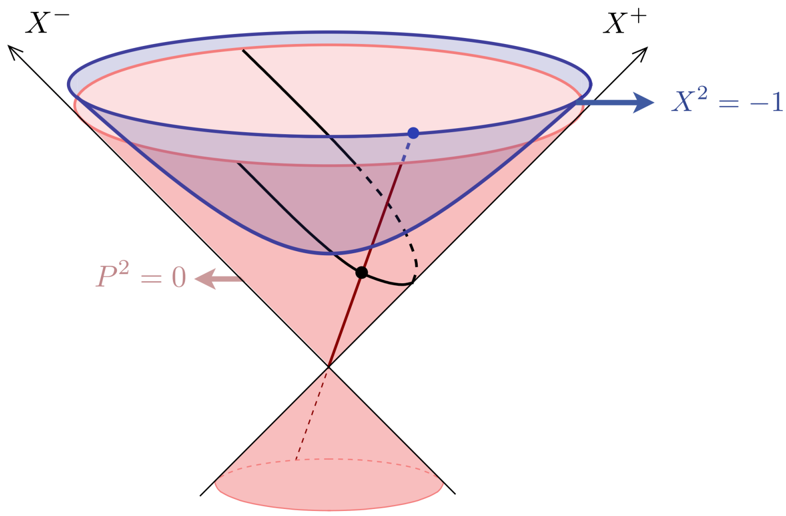

Figure 2.1: Euclidean AdS and its boundary in the embedding space. This picture shows the two-dimensional Euclidean AdS surface and the identification of a boundary point (in blue) with a light ray (in red) of the light cone, which intersects the Poincaré section on a (black) point. [20]

its isometry group is described by $SO(1, d+1)$. Therefore topologically, it is equivalent to a d+1-dimensional open ball which makes its conformal completion corresponding to the topological closure of this open ball. Hence the conformal boundary, given by the limit points of the $d+1$-dimensional open ball, is topologically equivalent to a d-dimensional sphere $\mathbb{S}^d$. One recalls, that in terms of the stereographic projection, the sphere $\mathbb{S}^d$ is homeomorphic to $\mathbb{R}^d \cup \{\infty\}$. This discussion allows us to access the conformal boundary of AdS by regarding it to $z = 0$ [42].

Consequently, a boundary formalism in Poincaré coordinates can be established by extending (2.5) to the boundary, i.e. it asymptotes to the light-cone (see Fig. 2.1 for a visualized understanding). The limit of this extension is ill-defined, thus we introduce the null projective cone coordinates P^A satisfying the embedding constraint (2.5) which are passing through the origin of $\mathbb{M}^{1,d+1}$, i.e.,

$$P^A P_A = 0, \quad P^A \simeq \lambda P^A \quad \text{for } \lambda \in \mathbb{R}, \lambda \neq 0. \tag{2.11}$$

After taking the limit $P^A \equiv \lim_{z\to 0} \left(z X^A L_{\mathrm{AdS}}^{-1} \right)|_{x^a = p^a}$, i.e., sending the radial coordinate towards the boundary of AdS, it provides the boundary coordinates [35], [30]

$$\begin{aligned} P^0 &= \frac{1}{2L_{\mathrm{AdS}}} \left(L_{\mathrm{AdS}}^2 + \delta_{ab} p^a p^b \right), \\ P^a &= p^a, \\ P^{d+1} &= \frac{1}{2L_{\mathrm{AdS}}} \left(L_{\mathrm{AdS}}^2 - \delta_{ab} p^a p^b \right), \end{aligned} \tag{2.12}$$

where the intrinsic boundary coordinate p^a is defined along the real numbers. Note that rescalings by λ have the effect of sending $z \to \lambda^{-1} z$, i.e., the boundary metric is rescaled by an overall factor causing a global dilatation. If some function $f(P^A)$ has scaling dimension Δ, then

$$f(\lambda P^A) = \lambda^{-\Delta} f(P^A). \tag{2.13}$$

A more detailed discussion to such rescalings will be addressed in the next section.

For the boundary coordinates, a metric can be induced which again is ill-defined in a direct way since the upper-half space Euclidean metric degenerates at $z = 0$. Nonetheless it is possible to define it by its limit

$$g_{\mu\nu} = \lim_{z\to 0} \frac{L^2_{\text{AdS}}}{z^2} \eta_{\mu\nu}, \tag{2.14}$$

with integration measure $dP \equiv dp$ on the boundary. Thus we have a conformal structure on the boundary providing a convenient setting for studying conformal field theories.

2.1.2 Invariant distances

To complete the perspective on the geometric aspects of the background, it is important to implement a notion of invariant distances in AdS defined in the ambient space.

By Synge's world function in the hyperbolic space $\sigma_{\mathbb{H}}$, we can define intrinsically the geodesic distance $s_{\mathbb{H}}$ in the Poincaré fundamental domain PAdS^2 between two arbitrary points, i.e.,

$$\sigma_{\mathbb{H}}(X, X') = \frac{1}{2} s_{\mathbb{H}}(X, X')^2. \tag{2.15}$$

Note that any intrinsically defined function can be expressed in terms of the embedding coordinates. One then uses Synge's world function in the ambient space $\sigma_{\mathbb{M}}$, which is globally defined in the form [1]

$$\sigma_{\mathbb{M}}(X, X') = \frac{1}{2} s_{\mathbb{M}}(X, X')^2 = \frac{1}{2} \eta_{AB} \left(X - X'\right)^A \left(X - X'\right)^B. \tag{2.16}$$

Here $s_{\mathbb{M}}$ is often referred to the chordal distance in the ambient space $\mathbb{M}^{1,d+1}$. Using the coordinate parametrization (2.7) provides a relation between intrinsic and ambient distances [21]:

$$\cosh\left(\frac{s_{\mathbb{H}}}{L_{\text{AdS}}}\right) = 1 + \frac{s^2_{\mathbb{M}}}{2L^2_{\text{AdS}}}. \tag{2.17}$$

In later discussions it will be convenient to make use of a dimensionless quantity u required to be invariant under the isometry group of Euclidean AdS $SO(1, d+1)$, thus reading in conformal coordinates [30]

$$u(X, X') \equiv \frac{s^2_{\mathbb{M}}}{L^2_{\text{AdS}}} = \frac{(z - z')^2 + \delta_{ab}(x - x')^a (x - x')^b}{zz'}. \tag{2.18}$$

We usually refer to it as the invariant chordal distance between two bulk points. Importantly, it can be extended to the conformal boundary as we send z' towards the boundary, i.e., $z' \to 0$, providing

$$\begin{aligned} u(X,P) \equiv \lim_{z'\to 0} \frac{z'}{L_{\text{AdS}}} u(X,X')\Big|_{x'^a=p^a} &= \frac{1}{L_{\text{AdS}} z}\left[z^2 + \delta_{ab}(x-p)^a(x-p)^b\right] \\ &= -2L_{\text{AdS}}^{-2} X\cdot P, \end{aligned} \tag{2.19}$$

defining an invariant distance of a point in the bulk to a boundary point. Thus between two boundary points we work out equivalently

$$\begin{aligned} u(P,Q) \equiv \lim_{z\to 0} \frac{z}{L_{\text{AdS}}} u(X,Q)\Big|_{x^a=p^a} &= \frac{1}{L_{\text{AdS}}^2}\delta_{ab}(p-q)^a(p-q)^b \\ &= -2L_{\text{AdS}}^{-2} P\cdot Q. \end{aligned} \tag{2.20}$$

Following the construction of a invariant distance intrinsically, we like to define its derivative. As discussed, AdS can be embedded into an ambient $d+1$-dimensional Minkowski space $\mathbb{M}^{1,d+1}$ where the induced squared line element on the hyperbolic space (2.8) can also be formed into

$$\mathrm{d}s^2_{\mathbb{H}^d} = \left(\eta_{AB} + L_{\text{AdS}}^{-2} X_A X_B\right)\mathrm{d}X^A\,\mathrm{d}X^B. \tag{2.21}$$

Thereon we obtain a covariant derivative in hyperbolic space by projecting the ambient space's partial derivative onto the hyperboloid, i.e., we make use of the projector

$$g_A{}^B \equiv \delta_A{}^B + L_{\text{AdS}}^{-2} X_A X^B, \tag{2.22}$$

providing in particular the derivatives

$$\begin{aligned} \nabla_A\varphi &= \left(\delta_A{}^B + L_{\text{AdS}}^{-2} X_A X^B\right)\partial_B\varphi, \\ \nabla^2\varphi &= \left(\eta^{AB} + L_{\text{AdS}}^{-2} X^A X^B\right)\partial_A\partial_B\varphi + 2L_{\text{AdS}}^{-2} X^A\partial_A\varphi. \end{aligned} \tag{2.23}$$

Acting on a function of invariant distances leads then to the expressions [30]

$$\begin{aligned} L_{\text{AdS}}^2\nabla^2 f(u(X,X')) &= u\,(u+4)\,f''(u) + 2\,(u+2)\,f'(u), \\ L_{\text{AdS}}^2\nabla^2 f(-2L_{\text{AdS}}^{-2}X\cdot P) &= \left(-2L_{\text{AdS}}^{-2}X\cdot P\right)^2 f'' + 2\left(-2L_{\text{AdS}}^{-2}X\cdot P\right) f'. \end{aligned} \tag{2.24}$$

It can be checked by hand that this connection is metric compatible, torsionless and gives the right curvature form of the considered AdS space. Hence we defined an unique Levi-Civita connection on this space.

2.2 Conformal Field Theory

An important class of field theories are conformal field theories (CFT's) which are invariant under the conformal group, which is an extension of the Poincaré group. Such an invariance gives rise to transformations preserving local angles but not necessarily distances, which results into imposing strong constraints on the dynamics of the theory.

In this section, we look at the conformal symmetry which is preserved in any CFT and afterwards discuss the consequences of this conformal invariance, especially focusing on the two- and three-point functions of CFT's, which are essential to provide a deeper understanding of how CFT's manifest in the AdS/CFT correspondence. At the end, we will briefly dive into the topic of logarithmic conformal field theories.

2.2.1 Conformal Symmetry

Consider a $p+q$-dimensional space $\mathbb{R}^{p,q}$ endowed with the metric tensor $\eta_{\mu\nu}$. We seek for transformations defined as an isometry by a change of coordinates $x^\mu \to x'^\mu$ preserving the form of the metric tensor up to a scale factor, i.e.,

$$g'_{\mu\nu}(x'^\mu) = \Lambda^2(x) g_{\mu\nu}(x), \tag{2.25}$$

where $\Lambda(x)$ is an arbitrary function of the coordinates called the conformal factor. Such transformations are commonly known as conformal transformations, displaying a generalization of ordinary symmetry transformations, in that every symmetry transformation satisfies the above metric tensor change by setting $\Lambda^2(x^\mu) = 1$ for all x^μ, i.e., the conformal transformation turns into a Poincaré transformation. The wording *conformal* implies that the angle between curves crossing at any point is preserved [33].

Conformal transformations form a group under composition, namely the conformal group. The representations of its algebra act on carrier spaces spanned by quasi-primary conformal operators and their descendants. A local scalar operator $\mathcal{O}(x^\mu)$ transforming under a conformal transformation as

$$\mathcal{O}_\Delta(x^\mu) \to \mathcal{O}'_\Delta(x'^\mu) = \left|\frac{\partial x'^\mu}{\partial x^\mu}\right|^{-\frac{\Delta}{d}} \mathcal{O}_\Delta(x^\mu), \tag{2.26}$$

is called a quasi-primary operator of scaling or conformal dimension Δ. The derivatives of a quasi-primary operator are not themselves again quasi-primary operators but rather called secondaries as well as descendants of the quasi-primary operator.

We briefly mention the consequences of the definition (2.26) for finite transformations $x^\mu \to x'^\mu = x^\mu + a^\mu$. A detailed investigation can be studied in [25]. The finite transformations are then given by

$$\begin{aligned}
&\text{(translation)} && x'^\mu = x^\mu + a^\mu \\
&\text{(dilation)} && x'^\mu = \lambda x^\mu \\
&\text{(lorentz)} && x'^\mu = \omega^\mu{}_\nu x^\nu \\
&\text{(SCT)} && x'^\mu = \frac{x^\mu + b^\mu x\cdot x}{1 + 2a\cdot x + (a\cdot a)(x\cdot x)}.
\end{aligned} \tag{2.27}$$

By assuming that the operators are unaffected by the transformation, we can define the generators of the conformal group to be

$$\begin{aligned}
&\text{(translation)} && P_\mu = -i\partial_\mu \\
&\text{(dilation)} && D = -ix^\mu\partial_\mu \\
&\text{(rotation)} && M_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \\
&\text{(SCT)} && K_\mu = -i\left(2x_\mu x^\nu\partial_\nu - \vec{x}^2\partial_\mu\right).
\end{aligned} \tag{2.28}$$

which obey the $SO(p+1,q+1)$ commutation relations

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right), \tag{2.29}$$

where $a,b,c,d \in \{0,1,\ldots,d+1\}$, if we redefine the generators to be

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu} \qquad & J_{\mu 1} &= \frac{1}{2}\left(P_\mu - K_\mu\right) \\ J_{21} &= D \qquad & J_{\mu 2} &= \frac{1}{2}\left(P_\mu + K_\mu\right), \end{aligned} \tag{2.30}$$

where $J_{\mu\nu} = -J_{\nu\mu}$ and $\mu,\nu \in \{0,1,\ldots,d-1\}$. Thus the total number of generators of the conformal group in $d = p+q$ dimensions can be split up into

$$\begin{aligned} d &\quad \text{(translation)} \\ \frac{d(d+1)}{2} &\quad \text{(rotation)} \\ 1 &\quad \text{(dilatation)} \\ d &\quad \text{(specialconformal)} \end{aligned} \tag{2.31}$$

which sum up correctly to $\frac{(d+1)(d+2)}{2}$ generators of $SO(p+1,q+1)$. Such conformal symmetry imposes stringent constraints on the form of correlation functions for quasi-primary operators. In short terms, the non-coincidence terms of two- and three-point functions are essentially fixed up to a constant whereas higher point functions depend on arbitrary functions, namely an-harmonic ratios. We will discuss their derivation in the upcoming subsection.

2.2.2 Constraints on Correlation Functions

As we have seen, CFT's contain some symmetry constraints, which heavily restrict the form of correlation functions to be as specific as it can only get. By the fact that the measure and action are conformally invariant, which is explained in more detail in [25], we are allowed to promote the transformation (2.26) to correlation functions. Thus for n primary fields $\mathcal{O}_{\Delta_1},\ldots,\mathcal{O}_{\Delta_n}$ we find

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(x_i^\mu) \right\rangle = \prod_{i=1}^{n} \left| \frac{\partial x'^\mu}{\partial x^\mu} \right|_{x^\mu = x_i^\mu}^{\frac{\Delta_i}{d}} \left\langle \prod_{i=1}^{n} \mathcal{O}_{\Delta_i}(x_i'^\mu) \right\rangle. \tag{2.32}$$

By utilizing the specific symmetries which are defined for CFT's, we can constrain more and more the form of the correlation functions, for instance in the case of the two-point function. First, we require by rotational and translational symmetry

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu)\mathcal{O}_{\Delta_2}(x_2^\mu)\rangle = f(|x_1^\mu - x_2^\mu|). \tag{2.33}$$

If we take into account the scale transformation $x^\mu \to \lambda x^\mu$, we then obtain

$$f(|x_1^\mu - x_2^\mu|) = \lambda^{\Delta_1+\Delta_2} f(\lambda\,|x_1^\mu - x_2^\mu|), \tag{2.34}$$

which fixed the two-point correlation function up to a constant coefficient

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu)\mathcal{O}_{\Delta_2}(x_2^\mu)\rangle = \frac{C}{|x_1^\mu - x_2^\mu|^{\Delta_1+\Delta_2}}. \tag{2.35}$$

What is left is the invariance under special conformal transformations, where we have

$$\left|\frac{\partial x'^{\mu}}{\partial x^{\mu}}\right| = \frac{1}{(1-2b^{\mu}x_{\mu}+b^2x^{\mu}x_{\mu})^d}. \tag{2.36}$$

Substituting it into (2.26) and using the covariance of (2.35) fixes $\Delta_1 = \Delta_2$, as it satisfies the constraint

$$\frac{C}{|x_1^{\mu}-x_2^{\mu}|^{\Delta_1+\Delta_2}} = \frac{C}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{|x_1^{\mu}-x_2^{\mu}|^{\Delta_1+\Delta_2}} \tag{2.37}$$

with $\gamma_i = \big(1-2b_{\mu}(x_i)^{\mu}+b^2(x_i)_{\mu}(x_i)^{\mu}\big)$. Thus we are left with two quasi-primary operators, which are correlated only if they happen to possess the same scaling dimension

$$\big\langle \mathcal{O}_{\Delta_i}(x)\mathcal{O}_{\Delta_j}(y)\big\rangle = \delta_{ij}\cdot\frac{C}{|x_1-x_2|^{\Delta_i+\Delta_j}}, \tag{2.38}$$

which is fixed up to a re-scaling and vanishes if the primary operators are assigned with different scaling dimensions.

These properties happen to be true for all correlation functions of local operators in a conformal field theory. Hence, a similar result for three-point functions is found for quasi-primary operators $\mathcal{O}_i$, $\mathcal{O}_j$, $\mathcal{O}_k$ as we apply the covariance under rotations, translations and dilatations to force it into the form

$$\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}^{(abc)}}{|x_1-x_2|^a\,|x_2-x_3|^b\,|x_1-x_3|^c}, \tag{2.39}$$

with $a+b+c = \Delta_i+\Delta_j+\Delta_k$. Again under special conformal transformations, we end up with a system of constraints for which unique solutions can be found to be

$$\begin{aligned} a &= \Delta_i+\Delta_j-\Delta_k \\ b &= \Delta_j+\Delta_k-\Delta_i \\ c &= \Delta_k+\Delta_i-\Delta_j. \end{aligned} \tag{2.40}$$

Hence, for three quasi-primary operators, we find:

$$\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}\,|x_2-x_3|^{\Delta_j+\Delta_k-\Delta_i}\,|x_1-x_3|^{\Delta_k+\Delta_i-\Delta_j}}, \tag{2.41}$$

which are completely determined by structure constants C_{ijk}.

For higher-point correlation functions, conformal symmetry is no longer sufficient to constrain the space dependence of the correlation functions. Thus, higher-point correlation functions are functions of the conformal invariants constructed out of the x_i, not determined by the conformal symmetry alone. Once we know the spectrum of primaries and their three-point functions, all correlation functions of local operators in a CFT are completely fixed due to the decomposition using the operator product expansion (OPE). It is a general property of local QFT's and particularly powerful in CFT's. In short terms, the product of two operators at nearby points can be rewritten as a series of operators at one point only:

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k C^k{}_{ij}(x-y)\mathcal{O}_k(y), \tag{2.42}$$

and conformal symmetry then fixes

$$C^k{}_{ij}(x-y) = C^k{}_{ij}\,|x-y|^{-(\Delta_i+\Delta_j-\Delta_k)}\,. \tag{2.43}$$

In unitary field theories, there exists a lower bound on the scaling dimension for primary operators, which is fixed in d-dimensional CFT's to be the unitarity bound [49]

$$\Delta \geq \max\left\{0, \frac{d-2}{2}\right\}. \tag{2.44}$$

In quantum field theories, unitarity is a fundamental requirement ensuring that the sum of the probabilities of all outcomes is conserved and that physical states have non-negative norms. Such states correspond to operators acting on the vacuum. Hence, the unitarity condition ensures also that the correlation functions behave properly at short distances and avoid pathologies.

2.2.3 Logarithmic CFT

There are special kinds of conformal field theories where unitarity can be given up, i.e., allowing the appearance of negative-norm states or indecomposable representations of the conformal algebra. In particular, as the representations turn out to be indecomposable, the operators form Jordan blocks by a change of basis, indicating that the theory does not have a simple diagonalizable structure for the conformal generators. In such cases the separation of spatial points in correlation functions can contain logarithmic dependence. Their emergence are commonly referred to logarithmic conformal field theories (LCFT). A good introduction can be found in [39].

A LCFT is characterized by the action of the dilatation operator D being non-diagonalizable [36], leading to logarithmic multiplets $\{\mathcal{O}_a\}_{a=1,\ldots,r}$ of rank r, where dilatations act as

$$\mathcal{O}_a(x) \to c^{-\Delta_{ab}}\mathcal{O}_b(x), \quad a = 1,\ldots,r. \tag{2.45}$$

The main difference is that the scaling dimension of the operator Δ_{ab} turns into a non-diagonalizable matrix, with all eigenvalues equal to the scaling dimension Δ of the multiplet. Thus it is possible to change the operator basis such that Δ_{ab} can be put into Jordan normal form

$$\Delta_{ab} = \begin{pmatrix} \Delta & 1 & & & 0 \\ & \Delta & 1 & & \\ & & \Delta & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \Delta \end{pmatrix}. \tag{2.46}$$

The size of the Jordan block then equals the rank of the logarithmic multiplet.

The most important aspect is how the correlation functions change in form for such LCFT's. In contrast to unitary CFT's, the dilatation and conformal generators D and K^μ now mix different operators in the multiplet:

$$\begin{aligned} D\mathcal{O}_a(x) &= \left[\Delta_a{}^b + \delta_a{}^b(x\cdot\partial)\right]\mathcal{O}_b(x), \\ K^\mu\mathcal{O}_a(x) &= 2x^\mu\left[\Delta_a{}^b + \delta_a{}^b(x\cdot\partial)\right]\mathcal{O}_b(x) - x^2\partial^\mu\mathcal{O}_a(x). \end{aligned} \tag{2.47}$$

Hence, by using the conformal Ward identities again, new constraints for correlation functions are imposed by dilatation and special conformal transformation. Generators corresponding to translations do not change in this setup, thus ensuring for correlation functions the dependence of squared spatial separation.

Consider the two-point function of operators in the same logarithmic multiplet, for instance of rank two

$$\langle\mathcal{O}_a(x_i)\mathcal{O}_b(x_j)\rangle := f_{ab}\left(|x_i - x_j|^2\right), \quad a,b = 1,2. \tag{2.48}$$

A solution of the applied conformal Ward identities upon the two-point function can be found. The two-point function of the logarithmic multiplet is [39]

$$f_{ab} = c\cdot|x_i - x_j|^{-2\Delta}\begin{pmatrix} -\log\left(\mu^2|x_i - x_j|^2\right) & 1 \\ 1 & 0\end{pmatrix}_{ab}. \tag{2.49}$$

In summary, two-point functions do contain logarithmic dependence on spatial separations. Note the appearance of a scale μ within the logarithm which can be removed by changing the operator basis.

2.3 AdS/CFT Correspondence

At the end of this chapter, we shall discuss the famous AdS/CFT correspondence. After looking at Anti-de Sitter spaces and Conformal Field Theories in an isolated sense, we are left with the question, in what sense a CFT is equivalent to a gravitational theory in AdS space. A possible answer to this issue did appear in the late 20th century, when Maldacena had conjectured in [42] the existence of a one-to-one correspondence between a Type IIB superstring theory on the background $AdS^5 \times S^5$ with coupling constant g_s and a $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four-dimensional with gauge group $SU(N)$ and coupling constant g_{YM} , i.e., they produce the same observables. In particular, in the large N limit, massive states decouple and string theories eventually can be approximated by (super)-gravity which can be regarded as a low energy limit of the string theory. If such a theory is weakly coupled, i.e. the space curvature is small, its quantum nature vanishes and the theory ends up to be classical. In order that such approximations hold, it is first necessary to study a strongly coupled system on the boundary and second we demand a large number of degrees of freedom.

Generally speaking, the paradigm of the correspondence states that there exists a connection between the fields describing quantum gravity in a $d+1$-dimensional AdS space, which often are referred to as bulk fields and operators in non-gravitational quantum field theories (CFT), which are defined on its d-dimensional boundary . For that reason, we simply refer it as the AdS/CFT correspondence [42], [38], [54]. Consequently, it let us assume that there must exist some dictionary between bulk fields in AdS and operators in a CFT, which provides the necessary formulas to translate objects from the theory of the interior into the theory living on its boundary.

Such a dictionary, which translates the objects between these two theories must be mathematically well-defined, which is the case if the isometries in AdS^{d+1} correspond to the conformal isometries in CFT^d [26]. For the case of Euclidean theories, they share a topological equivalence, which we discussed in the first section, as CFT's in a Euclidean context are defined naturally on S^d and the $d{+}1$-dimensional Euclidean Anti-de Sitter space is a $d{+}1$-dimensional hyperbolic space, therefore formally equivalent to the disk D^{d+1} with S^d as its boundary. Thus, the boundary of $\mathbb{E}\mathrm{AdS}^{d+1}$ is topologically equivalent to a Euclidean CFT. This prescription has been extensively discussed in [54].

Another identification can be made, by explicitly deriving the generators of the isometry group of $\mathbb{E}\mathrm{AdS}^{d+1}$ and send the coordinates towards the boundary. After the limit, we should be able to obtain the Poincaré coordinate system on the boundary, defined in (2.12). With good foresight, we will only investigate this identification in the two-dimensional case of the Euclidean AdS space. Hence its isometry group is $SO(1,2)$ with the Lie algebra $\mathfrak{su}(1,1)$ given by

$$\left[\Theta^a, \Theta^b\right] = i\varepsilon^{ab}{}_c\Theta^c, \tag{2.50}$$

where Θ^a are the generators of the Lie group $SO(1,2)$ given by

$$\begin{aligned} i\Theta^0 &= X^1\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X_1} \\ i\Theta^1 &= X^0\frac{\partial}{\partial X^2} - X_2\frac{\partial}{\partial X_0} \\ i\Theta^2 &= X_1\frac{\partial}{\partial X_2} - X^2\frac{\partial}{\partial X^1} \end{aligned} \tag{2.51}$$

Thus we are given three different generators, namely one translation, one dilatation and one special conformal transformation. After acting on coordinate functions, they are often referred to as the Killing vector fields generating isometries.

We can explicitly express the generators in terms of the Poincaré coordinates, after plugging in the transformations

$$\begin{aligned} \frac{\partial}{\partial X^0} &= \frac{\partial z}{\partial X^0}\frac{\partial}{\partial z} + \frac{\partial x}{\partial X^0}\frac{\partial}{\partial x} = -\frac{z}{L_{\mathrm{AdS}}^2}\left(z\frac{\partial}{\partial z} + x\frac{\partial}{\partial x}\right) \\ \frac{\partial}{\partial X^1} &= \frac{\partial z}{\partial X^1}\frac{\partial}{\partial z} + \frac{\partial x}{\partial X^1}\frac{\partial}{\partial x} = \frac{z}{L_{\mathrm{AdS}}}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial X^2} &= \frac{\partial z}{\partial X^2}\frac{\partial}{\partial z} + \frac{\partial x}{\partial X^2}\frac{\partial}{\partial x} = -\frac{z}{L_{\mathrm{AdS}}^2}\left(z\frac{\partial}{\partial z} + x\frac{\partial}{\partial x}\right), \end{aligned} \tag{2.52}$$

to

$$\begin{aligned} i\Theta^0 &= \frac{1}{2L_{\text{AdS}}}\left(L_{\text{AdS}}^2 - z^2 + x^2\right)\frac{\partial}{\partial x} + \frac{xz}{L_{\text{AdS}}}\frac{\partial}{\partial z} \\ i\Theta^1 &= -\left(z\frac{\partial}{\partial z} + x\frac{\partial}{\partial x}\right) \\ i\Theta^2 &= \frac{1}{2L_{\text{AdS}}}\left(L_{\text{AdS}}^2 + z^2 - x^2\right)\frac{\partial}{\partial x} - \frac{xz}{L_{\text{AdS}}}\frac{\partial}{\partial z}. \end{aligned} \tag{2.53}$$

Via the identities $\Theta^0 = \frac{1}{2}(P+K)$, $\Theta^1 = D$ and $\Theta^2 = \frac{1}{2}(P-K)$ we find the corresponding generators of the transformation, that correspond in the same way as we would have put it into a conformal context:

$$iP = L_{\text{AdS}}\frac{\partial}{\partial x}, \qquad iD = -x\frac{\partial}{\partial x}, \qquad iK = \frac{1}{L_{\text{AdS}}}\left[\left(x^2 - z^2\right)\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z}\right]. \tag{2.54}$$

Thus we end up with corresponding commutation relations, like in [49]

$$[D,P] = P, \qquad [D,K] = -K, \qquad [K,P] = 2D. \tag{2.55}$$

We have fully described the isometry group of $\mathbb{E}\text{AdS}^2$ and obtain the Poincaré boundary coordinates in (2.12) by sending the radial coordinate z towards the boundary, i.e. $z \to 0$ in (2.53). Therefore we have shown, that the isometry group in $\mathbb{E}\text{AdS}^2$ changes to a conformal group, described by a one-dimensional Euclidean CFT. A deeper discussion can be found in [56].

Consider now the following setup. Let a scalar field φ be in the bulk satisfying the usual Klein-Gordon equation with boundary condition $\varphi = \varphi_0$ at the boundary of AdS space. Then there exists to each field φ_i a corresponding local scalar operator $\mathcal{O}^i$ in the boundary theory, where the appropriately re-scaled boundary value φ_0 of the bulk field φ acts as a source for the scalar operator. Hence, we are able to display the duality in terms of their generating functionals [43]

$$Z_{\text{CFT}}[\varphi_0] := \left\langle \exp\left(\int_{\partial AdS} \mathrm{d}^d\vec{x}\,\varphi_0(\vec{x})\mathcal{O}(\vec{x})\right)\right\rangle_{\text{CFT}} = Z_{\text{AdS}}[\varphi_0], \tag{2.56}$$

where $Z_{\text{AdS}}[\varphi_0]$ is the bulk partition functional as integral over all bulk field configurations $\varphi(\vec{x})$ which take the value φ_0 at the boundary of the theory, i.e.

$$Z_{\text{AdS}}[\varphi_0] = \int_{\varphi=\varphi_0} \mathcal{D}[\varphi]\,\exp\left(-S_{\text{AdS}}[\varphi]\right). \tag{2.57}$$

One postulates, by considering a weakly-coupled quantum field theory, the generating functional is dominated by the saddle points of the string theory, i.e. at tree level reducing it to a classical super-gravity action

$$Z_{\text{AdS}} \simeq \exp\left(-N^2 S_{\text{class}}[\varphi] + \mathcal{O}(\alpha')\right) + \mathcal{O}(g_s). \tag{2.58}$$

Hence, the bulk partition function on the right hand side of (2.56) can then be approximated by a classical Euclidean action

$$Z_{\text{CFT}}[\varphi_0] = \exp\left(-S_{\text{AdS}}^{\text{on-shell}}[\varphi]\right)\Big|_{\varphi=\varphi_0}. \tag{2.59}$$

Imposing specific boundary conditions for φ and defining the regularity at the horizon then leads to unique solutions to the equation of motion [24], [3].

Thus, in the classical limit the correspondence states that the classical gravity action is the generating functional of connected correlation functions in the CFT. The correlation functions of a theory encode all of its information.

As a result, the connected n-point conformal correlation function can be computed by the generating function

$$W := \log\left(Z_{\text{bulk}}[\varphi_0]\right) = S_{\text{bulk}}^{\text{on-shell}}[\varphi]\big|_{\varphi=\varphi_0}\,, \tag{2.60}$$

of the gravity theory

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \left.\frac{\delta^{(n)} S_{\text{bulk}}^{\text{on-shell}}}{\delta\varphi_0(x_1)\dots\delta\varphi_0(x_n)}\right|_{\varphi_0=0}. \tag{2.61}$$

We understand the classical Euclidean action $S_{\text{bulk}}^{\text{on-shell}}$ to be renormalized with its counterterms included. This is necessary since the action diverges in general because the volume of AdS is infinite and the induced metric on the boundary receives a double pole. One way to regulate the gravity action is by restricting the bulk integral to the domain $z > \epsilon > 0$ and evaluating the boundary integral at $z = \epsilon$. After adding the counterterms the limit $\epsilon \to 0$ can be taken safely.

Since we are working at an energy/length scale at which the bulk space is AdS, i.e., asymptotically, the boundary theory behaves like a CFT. If then conformal transformations of the boundary commute with the boundary Hamiltonian, the operator dual to the bulk field must transform in some irreducible representation of the boundary conformal group. Hence if coordinates are scaled, i.e.

$$x^\mu \to \lambda \cdot x^\mu, \quad \lambda > 0, \tag{2.62}$$

then the field φ will transform as

$$\varphi \to \lambda^{-\Delta}\varphi. \tag{2.63}$$

Here Δ is identified as the conformal dimension of φ, meaning that a field with positive Δ scales up with the energy and vice versa with the coordinates.

In a classical CFT, we observe conformal dimension of fields as their classical energy dimensions but in a quantum sense they need to be renormalized in general. This change in dimension of an operator due to quantum loops is known as its anomalous dimension. The bulk fields then approach these boundary fields by the relation

$$\lim_{z\to 0} z^{\Delta-d}\varphi(\vec{x}) = \mathcal{N}\varphi_0(\vec{x}), \tag{2.64}$$

where $\mathcal{N}$ denotes a normalization factor which will be determined throughout the thesis.

Notice the re-scaling factor $z^{\Delta-d}$ which is necessary to obtain a convergent boundary field. In particular, if the operator $\mathcal{O}$ has scaling dimension Δ the source φ_0 has to have scaling dimension $d-\Delta$ such that the integral $\int \mathrm{d}^d x\, \varphi_0 \mathcal{O}$ is invariant under conformal transformations. Hence by specifying boundary conditions for bulk fields, loosely speaking we can relate those boundary values to CFT correlation functions of local operators whose generating functional is constructed out of the source.

As we had mentioned, some kind of dictionary is supposed to exist between the theories. Therefore we stress this formalism between the masses of the bulk fields and the scaling dimensions of their operator. The mass-squared parameter m^2 of an AdS scalar field and the conformal dimension Δ of the corresponding quasi-primary scalar operator satisfy

$$\Delta\,(\Delta - d) = L_{\mathrm{AdS}}^2 m^2. \tag{2.65}$$

This is one of the key result of the continuum AdS/CFT correspondence since it is determined by the asymptotic boundary behavior of the solutions of the Klein-Gordon equation in AdS space. This is realized by conformally transforming AdS space into flat space, where the scalar field experiences a shift of its mass m^2 [5]. Hence, we say that the mass is fixed by the Breitenlohner-Freeman bound [11], [12] which marks a stable potential minimum for the fields. For that reason, even small negative masses are possible.

Thus for a given bulk mass m, there exist two solutions of (2.65) arising analytically, reading

$$\Delta_\pm = \frac{d}{2} \pm \nu; \qquad \nu = \sqrt{\frac{d^2}{4} + L_{\mathrm{AdS}}^2 m^2}, \tag{2.66}$$

where we identify the relation $\Delta_+ = d - \Delta_- \geq \Delta_-$. Which one is chosen is determined by the corresponding boundary condition. Thus, a solution to the Klein-Gordon equation, i.e. a free scalar field φ with bulk mass m behaves near the boundary as [24]

$$\varphi(x^\mu) \xrightarrow{z\to 0} z^{\Delta_+}\left(A_+(x^a) + \mathcal{O}(z^2)\right) + z^{\Delta_-}\left(\varphi_0(x^a) + \mathcal{O}(z^2)\right). \tag{2.67}$$

The chosen boundary condition then preserves the isometries of AdS. Thus it is required that the Euclidean action is a finite function of φ^i's, i.e. the modes are required to be normalizable. Therefore, for $L_{\mathrm{AdS}}^2 m^2 > 1 - \frac{d^2}{4}$, only the positive solution in (2.66) is possible for which only the Dirichlet boundary condition exists to fix φ_0, whereas in the range $-\frac{d^2}{4} \leq L_{\mathrm{AdS}}^2 m^2 \leq 1 - \frac{d^2}{4}$, two kinds of boundary conditions exist for a scalar field on AdS space [15]. Thus, we say that Δ_+ corresponds to the standard quantization of the field with φ_0 acting as source and Δ_- corresponds to the alternative quantization where A is acting as source with dimension Δ_- [24]. For that reason, we are able to relate the unitarity bound (2.44) defined in a d-dimensional CFT to the Breitenlohner-Freeman bound.

We can conclude this chapter by defining the n-point correlation function of local scalar operators in the boundary CFT and bring it into correspondence of boundary values of the scalar fields in the bulk theory to be [17]

$$\langle \mathcal{O}_{\Delta^+}(\vec{x}_1) \cdots \mathcal{O}_{\Delta^+}(\vec{x}_n)\rangle^{\mathrm{BCFT}} = \lim_{z_1\to 0} z_1^{\Delta_+} \cdots \lim_{z_n \to 0} z_n^{\Delta_+} \langle \varphi_0(\vec{x}_1) \cdots \varphi_0(\vec{x}_n)\rangle^{\mathrm{bulk}}. \tag{2.68}$$

We properly introduced the physical models and their relation in the context of the AdS/CFT correspondence. It is important to note that we have not defined yet what a quantum field theory on the AdS space looks like, thus we can not assign any meaning into the physical properties of the bulk fields. We only drew the canonic picture in which the correspondence should appear. As we will begin to analyze it, the next chapter will provide a meaning to the bulk fields, for which we want to develop a viable dictionary.

Chapter 3

Scalar Fields and Propagators

After obtaining some insights of the AdS/CFT correspondence, we like to construct a quantum field theory in the given framework. This chapter will focus first on studying the behavior of a scalar field in $\mathbb{E}\mathrm{AdS}^2$, then aiming to construct the propagators that govern the quantum field theory within the AdS/CFT context. At the end, we apply a regularization on the two-point correlation function in the free theory via spectral representations of the propagator itself.

3.1 Scalar fields in $\mathbb{E}\mathrm{AdS}^2$

Consider the action of a real massive scalar field $\varphi : \mathrm{PAdS}^2 \to \mathbb{R}$ living in the Poincaré fundamental domain of a two-dimensional Euclidean Anti-de Sitter space $\mathbb{E}\mathrm{AdS}^2$ [26], [43]

$$S[\varphi] = \frac{1}{2} \int_{\mathbb{E}\mathrm{AdS}_2} \mathrm{d}^2\vec{x}\, \sqrt{g} \left(g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m^2 \varphi^2 \right), \tag{3.1}$$

where m is the mass of the scalar field. Here we might define the mass in a way that it differs from the effective mass $m_0^2 - 2\xi L_{\mathrm{AdS}}^{-2}$, where ξ is the scalar-curvature coupling constant [3]. Applying the Euler-Lagrange equation to the action

$$\begin{aligned} \frac{\partial}{\partial \varphi} \left(m^2 \varphi^2 \right) &= \partial_\mu g^{\lambda\sigma} \frac{\partial}{\partial(\partial_\mu \varphi)} \left(\partial_\lambda \varphi \partial_\sigma \varphi \right) \\ \Leftrightarrow m^2 \varphi &= \partial_\mu \partial^\mu \varphi, \end{aligned} \tag{3.2}$$

leads to the equation of motion

$$\left(\Delta_{\mathrm{PAdS}^2} - m^2 \right) \varphi = 0. \tag{3.3}$$

where Δ denotes the invariant Laplace operator, which we will write in the form of the Laplace-Beltrami operator, i.e.

$$\Delta \equiv \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \right). \tag{3.4}$$

Notice that the operator transforms with respect to the Poincaré fundamental domain.

Theorem 3.1.1. *The Laplace-Beltrami operator transforms under* PAdS^2 *to*[1]

$$\Delta_{\mathrm{PAdS}^2} = \frac{z^2}{L_{\mathrm{AdS}}^2}\left(\partial_z^2 + \partial_x^2\right). \tag{3.6}$$

Proof. We perform the calculation via the corresponding covariant derivative for a covector field

$$g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \Gamma^\rho{}_{\mu\nu}\frac{\partial}{\partial x^\rho}\right).$$

Since the metric is symmetric and only has entries on the diagonal, we set $\mu = \nu$ and sum over $\mu = \rho \in \{z, x\}$ in order to evaluate the corresponding Christoffel symbols $\Gamma^\rho{}_{\mu\nu}$ with the formula

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial}{\partial x^\nu}g_{\sigma\mu} + \frac{\partial}{\partial x^\mu}g_{\sigma\nu} - \frac{\partial}{\partial x^\sigma}g_{\mu\nu}\right).$$

Thus, we apply all results to display the Laplace-Beltrami operator

$$\begin{aligned}\Delta_{\mathrm{PAdS}^2} &= g^{zz}\partial_z^2 + g^{xx}\partial_x^2 - g^{zz}\left(\underbrace{\Gamma^z{}_{zz}}_{-z^{-1}}\partial_z + \underbrace{\Gamma^x{}_{zz}}_{0}\partial_x\right) - g^{xx}\left(\underbrace{\Gamma^z{}_{xx}}_{z^{-1}}\partial_z + \underbrace{\Gamma^x{}_{xx}}_{0}\partial_x\right)\\ &= \frac{z^2}{L_{\mathrm{AdS}}^2}\left(\partial_z^2 + \partial_x^2\right) + \frac{z}{L_{\mathrm{AdS}}^2}\partial_z - \frac{z}{L_{\mathrm{AdS}}^2}\partial_x\\ &= \frac{z^2}{L_{\mathrm{AdS}}^2}\left(\partial_z^2 + \partial_x^2\right).\end{aligned}$$

□

We utilize the equivalence between the Poincaré fundamental domain and the two-dimensional hyperbolic space to translate a solution φ for the equation of motion (3.6) into the scalar field $\Phi : \mathbb{H}^2 \to \mathbb{R}$. After inserting (3.3) into the equation of motion let us find the partial differential equation [43]

$$\left(\partial_z^2 + \partial_x^2 - \frac{L_{\mathrm{AdS}}^2 m^2}{z^2}\right)\Phi = 0. \tag{3.7}$$

The given Klein-Gordon equation in PAdS^2 has transformed into a wave equation on $\mathbb{H}^2$ with a potential, singular at $z = 0$. The endpoint $z = 0$ is singular and limit-point where our uniquely determined solution is square integrable near the origin. As we are dealing with a maximally symmetric solution to Einstein's

[1] The Laplacian can also be given by

$$\Delta_{\mathrm{PAdS}^2} = -\frac{1}{L_{\mathrm{AdS}}^2}\left(-\Theta_0^2 + \Theta_1^2 + \Theta_2^2\right), \tag{3.5}$$

where Θ^a are the generators of the Lie group $SO(1,2)$, like in (2.53) or [56].

equation, the Fourier transformation exists but the singular behavior restricts us from performing a Fourier transformation along the z-direction since it is not well-defined on its domain [21]. Therefore we are only allowed to take Fourier transformations along the directions orthogonal to z, leading to

$$\Phi(\vec{x}) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{\sqrt{2\pi}} \, e^{ik\vec{x}} \hat{\Phi}_k(z). \tag{3.8}$$

where k is the momentum one-vector. Inserting its Fourier transformation in (3.7) encodes the differential equation living in Fourier space

$$\left[\partial_z^2 - \left(k^2 + \frac{L_{\text{AdS}}^2 m^2}{z^2}\right)\right] \hat{\Phi}_k(z) = 0. \tag{3.9}$$

Before we solve the given equation, we like to emphasize its structure.

Definition 3.1.2. *Let $\mathcal{L}$ be the Sturm-Liouville operator, a linear differential operator acting on distributions on $z \in (0, \infty)$ with spectral parameter $-k^2 < 0$, of the form*

$$\mathcal{L} = \left[\partial_z^2 - \left(k^2 + \frac{L_{\text{AdS}}^2 m^2}{z^2}\right)\right]. \tag{3.10}$$

Thereon we identify (3.9) by $\mathcal{L}\hat{\Phi}_k = 0$, which is a second-order linear inhomogeneous ordinary differential equation. Hence we are dealing with a singular Sturm-Liouville equation on $z \in (0, \infty)$ which can be brought into the form of a Bessel differential equation with its two linearly independent solutions

$$\hat{\Phi}_k(z) = (kz)^{\frac{1}{2}} \left[c_1 I_\nu(kz) + c_2 K_\nu(kz)\right], \tag{3.11}$$

where I_ν and K_ν are the modified Bessel functions of the first and second kind, respectively. Such solutions are by definition continuous in z and symmetric. The parameter ν arises from the Bessel equation which naturally connects the solutions to the masses of the bulk fields in AdS by

$$\nu^2 = \frac{1}{4} + L_{\text{AdS}}^2 m^2. \tag{3.12}$$

We recall the equivalence of the mass of bulk fields and the scaling dimension of operators (2.66). W.l.o.g. we then assume that $\nu \in (1, \infty)$ which corresponds to $L_{\text{AdS}}^2 m^2 \in \left[\frac{3}{4}, \infty\right)$.

Imposing the property that the solutions in (3.11) need to be regular everywhere in the interior and even for $z \to \infty$. By using the asymptotic form for modified Bessel functions,

$$I_\nu(z) \sim e^{kz}, \qquad K_\nu(z) \sim e^{-kz}, \tag{3.13}$$

we see that I_ν is very badly behaved, i.e. it diverges in the interior, thus $c_1 = 0$ in (3.11). On the other hand, the modified Bessel function of the second kind K_ν is well-behaved in the interior and non-normalizable at the boundary at infinity [3]. We neglect the momentum term and by the asymptotic behavior at the boundary $z \to 0$ we know that the solutions are power-law behaved

$$\Phi \sim z^{\Delta_\pm}, \qquad \Delta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + L_{\text{AdS}}^2 m^2}, \tag{3.14}$$

which resembles the form in (2.67), where we discussed the corresponding boundary conditions which ultimately depend on the integrability of the solution near the singular point.

The first desired result points to a boundary propagator in Anti-de Sitter space since it is expected to be equal to a two-point correlation function on the boundary CFT. Therefore a well-established start is to construct an Anti-de Sitter invariant propagator and then make use of the geometric construction of the boundary in Anti-de Sitter space to access the boundary CFT, which should be of equivalent form to any CFT at least shown for two-point functions.

3.2 Free massive bulk propagator

In this section we like to discuss the construction of the free massive propagator on two-dimensional Euclidean Anti-de Sitter spaces, i.e., the two-point function for some ground state, which must be invariant under all isometries of PAdS^2 and solution of the equation of motion in both entries [21].

For the discussion, we require the equation of motion of the covariant Green's function G_m with respect to the underlying background geometry PAdS^2. Hence we start by looking for solutions satisfying [17]

$$\left(\Delta_{\mathrm{PAdS}_2} - m^2\right) G_m(\vec{x};\vec{x}') = \frac{1}{\sqrt{g}}\delta(z-z')\delta(x-x'), \tag{3.15}$$

and boundary condition $G_m(\vec{x},\vec{x}')|_{\vec{x}\in\partial\mathrm{PAdS}^2} = 0$, where δ denotes the Dirac delta function and g is the determinant of the metric tensor, given explicitly by

$$g \equiv \det g_{\mu\nu} = \left(\frac{L_{\mathrm{AdS}}}{z}\right)^4. \tag{3.16}$$

We immediately identify the Green's function as the right inverse of the operator $\Delta_{\mathrm{PAdS}^2} - m^2$. By explicitly implementing transformations (3.6) on the operator, the expression (3.15) rewrites to

$$\left[\partial_z^2 + \partial_x^2 - \frac{L_{\mathrm{AdS}}^2 m^2}{z^2}\right] G_m(\vec{x};\vec{x}') = \delta(\vec{x}-\vec{x}'). \tag{3.17}$$

We perform a dimensional analysis to check if the given operator matches the dimension of the Dirac delta function. Indeed, since the propagator has to be the dimension of the length, we suggest that the $L_{\mathrm{AdS}}^2 m^2$ term is dimensionless in order to ensure dimensional consistency. The picture of L_{AdS} being the AdS radius and m having dimension of inverse length confirms the interpretation of m as a mass term appropriately scaled by L_{AdS}.

In [21] it is stated, that the Green's function in PAdS^2 is radially symmetric in its only spatial direction. This is the exact same observation which we made for the scalar field in the previous section. Thus we can define the Fourier transformation orthogonal to the radial direction z in the same way to be

$$G_m(\vec{x},\vec{x}') = \int_{\mathbb{R}} \frac{\mathrm{d}k}{\sqrt{2\pi}}\, e^{ik|x-x'|}\hat{G}_k(z,z'). \tag{3.18}$$

We can transform the given integral by implementing spherical coordinates to be the Fourier-Bessel transformation [22]

$$\begin{aligned}\int_{\mathbb{R}} \frac{\mathrm{d}k}{\sqrt{2\pi}}\, e^{ik|x-x'|}\hat{G}_k(z,z') &= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{d}k\, \hat{G}_k(z,z') \int_0^\pi \mathrm{d}\theta\, \sin^{-1}(\theta) e^{ik|x-x'|\cos(\theta)} \\ &= |x-x'|^{\frac{1}{2}} \int_0^\infty \mathrm{d}k\, k^{\frac{1}{2}} J_{-\frac{1}{2}}(k|x-x'|)\hat{G}_k(z,z').\end{aligned} \tag{3.19}$$

By (3.17), we can identify $\hat{G}_k$ applied with the Sturm-Liouville operator $\mathcal{L}$ to be the Dirac delta function in the radial direction z. Therefore we can use the closure equation

$$\delta(z-z') = \sqrt{zz'}\int_0^\infty \mathrm{d}p\, p J_\nu(pz) J_\nu(pz'), \tag{3.20}$$

to obtain

$$G_m(\vec{x};\vec{x}') = \frac{(zz'|x-x'|)^{\frac{1}{2}}}{\pi}\int_0^\infty \mathrm{d}p\, p^{\frac{1}{2}} K_{-\frac{1}{2}}(p|x-x'|) J_\nu(pz) J_\nu(pz'). \tag{3.21}$$

One can perform the integral after abusing the formula [37, 6.578, Eq.6]

$$\int_0^\infty \mathrm{d}p\, p^{\frac{1}{2}} K_{-\frac{1}{2}}(p|x-x'|) J_\nu(pz) J_\nu(pz') = (2\pi zz'|x-x'|)^{-\frac{1}{2}}\, Q^0_{\nu-\frac{1}{2}}(u), \tag{3.22}$$

where $2zz'u = |x-x'|^2 + z^2 + z'^2$, $|x-x'| > 0$ and $\mathrm{Re}(\nu) > -1$ and identify the Legendre function $Q^0_{\nu-\frac{1}{2}}$ in terms of a hypergeometric function by

$$Q^0_{\nu-\frac{1}{2}}(z) = \frac{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}{2^{\nu+\frac{1}{2}}\Gamma(\nu+1)} z^{-\nu-\frac{3}{2}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{4}; \nu+1; \frac{1}{z^2}\right). \tag{3.23}$$

Therefore applying the above mathematical formulas gives us a closed form representation of an $\mathbb{E}\mathrm{AdS}^2$-invariant massive propagator in position space (see also [15], [14])

$$G_m(\vec{x};\vec{x}') = \frac{\Gamma\left(\frac{1}{2}+\nu\right)}{2\pi^{\frac{1}{2}}\Gamma(1+\nu)} u^{-\left(\frac{1}{2}+\nu\right)}\, {}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}+\nu; 1+2\nu; -\frac{4}{u}\right). \tag{3.24}$$

Here ${}_2F_1(a,b;c;z)$ is a hypergeometric function and u is defined like in (2.18) by

$$u(\vec{x},\vec{x}') = \frac{(x-x')^2 + (z-z')^2}{zz'}. \tag{3.25}$$

Note that u vanishes if the points coincide, i.e. $\vec{x} = \vec{x}'$ and the hypergeometric function in (3.24) can be singular at $-4u^{-1} = 0, 1, \infty$ [22].

It will turn out very useful if we express (3.24) in terms of the scaling dimension $\Delta_+ = \frac{1}{2} + \nu$ by the means of the AdS/CFT correspondence, i.e.[2] [30]

$$G_{\Delta_+}(X,X') = \frac{\Gamma\left(\Delta_+\right)}{2\pi^{\frac{1}{2}}\Gamma\left(\Delta_+ + \frac{1}{2}\right)} u(X,X')^{-\Delta_+}\, {}_2F_1\left(\Delta_+, \Delta_+; 2\Delta_+; -\frac{4}{u(X,X')}\right), \tag{3.26}$$

[2] We will abuse the notation of the ambient space coordinates, in which the propagators can also be dependable by transforming the invariant chordal distance u appropriately. Thus we explicitly omit the notation of dependence in u.

which is referred in the following as the bulk propagator of a scalar field in PAdS^2.

3.3 Boundary propagators

Since we want to relate objects on the boundary of our background space $\mathbb{E}$AdS^2 with objects in the bulk theory, it is of utmost interest to derive propagators which either prescribe interactions between objects in the bulk and on the boundary or simply between objects, solely living on the boundary. The former is known as the bulk-to-boundary propagator and the latter will be classified as the boundary propagator itself.

Similar to the construction of Poincaré coordinates on the boundary, where we sent the radial components to the boundary under rescaling, the propagators we like to derive follow the same procedure. Hence, the bulk-to-boundary propagators take a very simply form in ambient space [35].

Theorem 3.3.1 (Bulk-to-boundary propagator). *For a scalar of mass $L_{\mathrm{AdS}}^2 m^2 = \Delta(\Delta - 1)$, the bulk-to-boundary propagator is given by the limit*

$$G_{\partial\Delta}(X,P) \equiv \lim_{z'\to 0} \left(\frac{z'}{L_{\mathrm{AdS}}}\right)^{-\Delta} G_\Delta(X,X') = c_\Delta \left(-2L_{\mathrm{AdS}}^{-2} X\cdot P\right)^{-\Delta}, \quad (3.27)$$

where the normalization constant

$$c_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{1}{2}}\Gamma\left(\Delta + \frac{1}{2}\right)}, \quad (3.28)$$

is fixed by consistency with the corresponding bulk propagator [7, Section 3.6].

Proof. To obtain the bulk-to-boundary propagator, we need to locate our source at the boundary of the space, i.e., we take the limit $z' \to 0$ and add a scale factor coming from the definition of the embedding coordinates at the boundary.

Using the expression for the bulk propagator in (3.26) gives

$$\begin{aligned} G_{\partial\Delta}(X,P) &\equiv \lim_{z'\to 0} \left(\frac{z'}{L_{\mathrm{AdS}}}\right)^{-\Delta} G_\Delta(X,Y) \\ &= \frac{\Gamma(\Delta)}{2\pi^{\frac{1}{2}}\Gamma\left(\Delta+\frac{1}{2}\right)} \lim_{z'\to 0}\left(\frac{L_{\mathrm{AdS}}z}{(z-z')^2+(x-x')^2}\right)^{\Delta} \\ &\qquad \times {}_2F_1\left(\Delta,\Delta;2\Delta;-\frac{4zz'}{(z-z')^2+(x-x')^2}\right). \end{aligned}$$

The limit of the hypergeometric function ${}_2F_1(a,b;c;z)$ can be evaluated by expanding it into its power series, thus it yields the analysis

$$\begin{aligned} \lim_{z\to 0} {}_2F_1(a,b;c;z) &= \lim_{z\to 0} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n} \\ &= \lim_{z\to 0}\left[1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2} + \cdots\right] \\ &= 1, \end{aligned}$$

valid for $z \in \mathbb{C}$ satisfying $|z| < 1$ and $(a)_n$ denotes the Pochhammer symbol. Therefore we obtain

$$\begin{aligned} G_{\partial\Delta}(X,P) &= \frac{\Gamma(\Delta)}{2\pi^{\frac{1}{2}}\Gamma\left(\Delta+\frac{1}{2}\right)}\left(\frac{L_{\mathrm{AdS}}z}{z^2+(x-x')^2}\right)^{\Delta} \\ &= \frac{\Gamma(\Delta)}{2\pi^{\frac{1}{2}}\Gamma\left(\Delta+\frac{1}{2}\right)}(-2L_{\mathrm{AdS}}^{-2}X\cdot P)^{-\Delta}, \end{aligned}$$

which shows the assumption. □

The bulk-to-boundary propagator is a homogeneous solution of the hyperbolic Klein-Gordon equation in PAdS^2

$$\left(\Delta_{\mathrm{PAdS}^2} - m^2\right) G_{\partial\Delta}(X,P) = 0, \tag{3.29}$$

and follows the necessary singular behavior while approaching asymptotically the boundary [35], [26]

$$\lim_{z\to 0}\left(z^{\Delta-1}G_{\partial\Delta}(X,P)\right) = \frac{1}{2\Delta-1}\delta(P-Q). \tag{3.30}$$

After we sent one of the bulk points to the boundary of the corresponding Euclidean AdS space, we like to do the exact same procedure for the second bulk point. Therefore the boundary propagator is given by the limit

$$G_{\partial\Delta}(P,Q) \equiv \lim_{z\to 0}\left(\frac{z}{L_{\mathrm{AdS}}}\right)^{-\Delta} G_{\partial\Delta}(X,Q) = C_{\Delta}\left(-2L^{-2}P\cdot Q\right)^{-\Delta}. \tag{3.31}$$

This result perfectly identifies with the expected expression of a CFT two-point correlation function.

3.4 Spectral Representation and Regularization

Analyzing the explicit formula for the bulk propagator, we notice a UV divergence appearing when the coinciding point limit is taken. Such singularities need to be regularized, for example by introducing certain cutoffs in a characteristic energy scale upon the bulk propagator to hide the infinities like in the case of a coinciding point limit as it generally happens to be divergent. A more detailed explanation of this phenomena can be studied in [47]

For that reason, the chosen regularization scheme is set to be the heat kernel method to arrange such cutoffs. As the explicit representation of the bulk propagator is not favorably, it is convenient to harmonically analyze the hyperbolic space, i.e. we develop a spectral approach which decomposes bulk propagators $G_{\Delta}(X,X')$ into bi-tensorial AdS harmonic functions $\Omega_{\nu}(X,X')$ with $\nu \in \mathbb{R}$ [35].

We begin the construction by using the isometries of AdS to consider a $SO(1,2)$ invariant, square integrable function $F(X,X')$ of two bulk points in Euclidean AdS. Thus it is a function of only one non-negative real variable [15]. One can then expand functions F into the integral form, [46, Appendix B], [19, Section 3.2], [45]

$$F(X,X') = \int_{-\infty}^{\infty} \mathrm{d}\nu\, f(\nu)\Omega_{\nu}(X,X'). \tag{3.32}$$

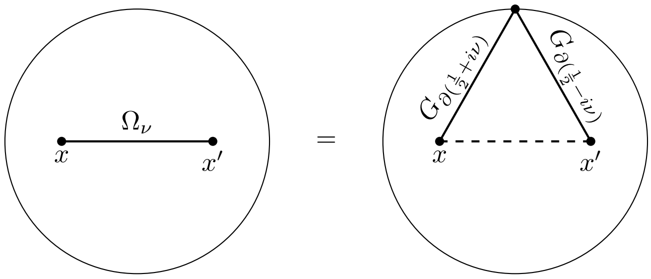

Figure 3.1: The split representation of the harmonic function obtained by integrating over the boundary point. The two circles represent the hyperbolic space as Poincaré disks where a straight line is a geodesic. The illustration is inspired by [35].

with the transform $f(\nu) = f(-\nu)$ as Ω_ν is assumed to be an even function of ν. Additionally, harmonic functions Ω_ν form a continuous basis of regular divergence-free eigenfunctions of the Laplacian operator [15], satisfying the equation of motion[3]

$$\left[L_{\mathrm{AdS}}^2 \Box_X + \left(\nu^2 + \frac{1}{4}\right)\right] \Omega_\nu(X, X') = 0. \tag{3.33}$$

Harmonic functions are normalized and also form a orthonormal set with respect to the standard inner product on $\mathbb{H}^2$, satisfying the normalization condition

$$\int_{-\infty}^{\infty} \mathrm{d}\nu \, \Omega_\nu(X, X') = L_{\mathrm{AdS}}^2 \delta(X, X'). \tag{3.34}$$

The latter can be checked later on by applying the Laplacian operator to a spectral representation of the bulk propagator (3.32) with given $f(\nu)$ [45, Appendix 4.C]. We denote (3.32) as spectral integrals diverging for large values of ν which we identify as UV divergence, thus providing a more appropriate expression of the bulk propagator by disentangling UV and IR bulk divergences. Hence they display a natural generalization of momentum space integrals in flat space to AdS encoding the symmetries of AdS [14].

A very advantageous definition of the harmonic function is the factorization into bulk-to-boundary propagators integrated over a common boundary point in AdS

$$\Omega_\nu(X, X') = \frac{\nu^2}{L_{\mathrm{AdS}}\pi} \int \mathrm{d}P \, G_{\partial(\frac{1}{2}+i\nu)}(X, P) G_{\partial(\frac{1}{2}-i\nu)}(X', P), \tag{3.35}$$

which is illustrated in figure 3.1. Its application leads to the decomposition of loop diagrams reducing their computation to boundary conformal integrals arising from the gluing of the tree-level bulk diagrams, and a spectral integral in the parameters ν [35, Section 1.1].

We immediately check that the split representation (3.35) does satisfy the eigenvalue equation (3.33) by applying the second order covariant derivative on

[3]The subscript on differential operators signifies that the derivative is begin taken with respect to X.

a function of invariant distances, i.e., using the identity (2.24) while setting $f(-2L_{\text{AdS}}^2 X \cdot P) := G_{\partial(\frac{1}{2}+i\nu)}(X,P)$ to see that

$$\begin{aligned}\Box_X \Omega_\nu(X,X') &= \frac{\nu^2}{L_{\text{AdS}}\pi} \int \mathrm{d}P\, \Box_X G_{\partial(\frac{1}{2}+i\nu)}(X,P) G_{\partial(\frac{1}{2}-i\nu)}(X',P) \\ &= -\frac{\nu^2}{L_{\text{AdS}}\pi} \int \mathrm{d}P\, \frac{\nu^2+\frac{1}{4}}{L_{\text{AdS}}^2} G_{\partial(\frac{1}{2}+i\nu)}(X,P) G_{\partial(\frac{1}{2}-i\nu)}(X',P) \\ &= -\frac{\nu^2+\frac{1}{4}}{L_{\text{AdS}}^2}\Omega_\nu(X,X').\end{aligned}$$

Consequently, the Euclidean AdS propagator belongs to the class of functions defined above in the standard quantization regime [45, Appendix 4.C], hence we employ the spectral representation of bulk propagators for scalar fields.

Theorem 3.4.1. *The harmonic space representation of the bulk propagator in* $\mathbb{E}\text{AdS}^2$ *with* $\Delta = \Delta_+ > \frac{1}{2}$ *is well-defined and given by [15], [14]*

$$G_\Delta(X,X') = \int_{-\infty}^{\infty} \mathrm{d}\nu \left[\nu^2 + \left(\Delta - \frac{1}{2}\right)^2\right]^{-1} \Omega_\nu(X,X'), \tag{3.36}$$

with harmonic functions defined in terms of bulk propagators evaluated at complex values of scaling dimension [4]

$$\Omega_\nu(X,X') \equiv \frac{i\nu}{2\pi}\left[G_{\frac{1}{2}+i\nu}(X,X') - G_{\frac{1}{2}-i\nu}(X,X')\right]. \tag{3.37}$$

Proof. We prove the harmonic representation of the bulk propagator by performing the integral (3.36) and using the expression of the harmonic kernels (3.37). Thus, we should end up with the bulk propagator again.

Performing the integral can be done via closing the contour in an appropriate region of the ν-plane and then summing over the included poles. First, consider the integration

$$I = \int_{-R}^{R} \frac{\mathrm{d}\nu}{2\pi} \frac{i\nu}{\nu^2 + (\Delta - \frac{1}{2})^2} \left[G_{\Delta_+}(X,X') - G_{\Delta_-}(X,X')\right].$$

We immediately see, that the integral can be split into two integrals $I = I_1 + I_2$. The leading behavior of the propagator at large Δ with fixed u is [15]

$$\sim \left(\sqrt{\frac{L_{\text{AdS}}^2 u}{4} + 1} - \frac{L_{\text{AdS}}}{2}\sqrt{u}\right)^{\Delta},$$

implying that I_1 only closes the contour in the lower-half ν-plane, and I_2 only in the upper-half ν-plane. As the calculations of both integrals are very analogous, we omit the evaluation of I_2 and restrict the proof to the integral

$$I_1 = \int_{-R}^{R} \frac{\mathrm{d}\nu}{2\pi} \frac{i\nu}{\nu^2 + (\Delta - \frac{1}{2})^2} G_{\Delta_+}(X,X').$$

[4] The derivation of the linear combination of two bulk propagators has been studied in [20, Appendix C]

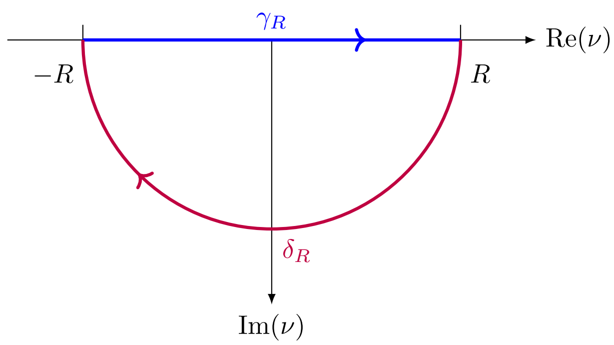

Figure 3.2: Contour in the lower half-plane, starting from $-R$ on the real axis, going to R on the real axis, and then following a half-circle in the lower half-plane, going clockwise from R to $-R$.

The only pole coming from the kernel contributing to the lower-half is $\nu = -i(\Delta_+ - \frac{1}{2})$. Looking at the propagator, the poles of $\Gamma(\Delta_+ + \frac{1}{2})$ get canceled out from poles of the hypergeometric function. Furthermore, for $\Delta_+ \in \mathbb{C}$, $\Gamma(\Delta_+)$ has simple poles at $\Delta_+ = -n$ for $n = 0, 1, \ldots,$ which are located in the upper-half, hence omitted. For $|z| < 1$, ${}_2F_1(\Delta_+, \Delta_+; 2\Delta_+; z)$ is analytic and has simple poles at $\Delta_+ = -\frac{n}{2}$ for $n = 0, 1, \ldots,$ which are located in the upper-half, hence omitted.

To perform the actual contour integration, we need to calculate the limit $R \to \infty$ to obtain our desired integral. We set

$$f(\nu) = \frac{1}{2\pi} \frac{iv}{\nu^2 + (\Delta - \frac{1}{2})^2} G_{\Delta_+}(X, X'),$$

and define contours

$$\begin{aligned} \gamma_R : [-R, R] &\to \mathbb{C} \\ t &\mapsto t, \end{aligned}$$

and

$$\begin{aligned} \delta_R : [\pi, 2\pi] &\to \mathbb{C} \\ \theta &\mapsto Re^{i\theta}, \end{aligned}$$

where $\Gamma_R := \gamma_R + \delta_R$ is identified as the complete contour of the integration (see Fig. 3.2). The notation yields

$$\oint_{\Gamma_R} \mathrm{d}\nu\, f(\nu) = \int_{\gamma_R} \mathrm{d}\nu\, f(\nu) + \int_{\delta_R} \mathrm{d}\nu\, f(\nu).$$

Left hand side can be calculated via Cauchy's residue theorem

$$\begin{aligned}\oint_{\Gamma_R} \mathrm{d}\nu\, f(\nu) &= -2\pi i \sum_{j=1}^{n} \operatorname{Res}(f,\nu_j)\\ &= -2\pi i \cdot \operatorname{Res}\left(\frac{i\nu}{2\pi}\frac{1}{\nu^2+(\Delta-\frac{1}{2})^2} G_{\Delta_+}(X,X'), -i(\Delta-\frac{1}{2})\right)\\ &= \lim_{\nu\to -i(\Delta-\frac{1}{2})} \frac{\nu}{\nu-i(\Delta-\frac{1}{2})} G_{\Delta_+}(X,X')\\ &= \frac{1}{2} G_\Delta(X,X')\end{aligned}$$

where ν_j are all poles covered by the contour integration. Therefore we need to show that

$$\int_{\delta_R} \mathrm{d}\nu\, f(\nu) \overset{R\to\infty}{\to} 0.$$

Using the parametrization $\nu = Re^{i\theta}$ and the estimation lemma gives

$$\left|\int_{\delta_R} \mathrm{d}\nu\, f(\nu)\right| \leq \sup_{\theta\in[\pi,2\pi]} \left|\frac{R^2 e^{i\theta}}{R^2 e^{2i\theta}+(\Delta-\frac{1}{2})^2} G_{\frac{1}{2}+iRe^{i\theta}}(X,X')\right| \overset{R\to\infty}{\to} 0.$$

In the last step, we used Stirling's formula as an approximation of Gamma functions appearing in the expression of the bulk propagator for large scaling dimension Δ_+ (for asymptotic behavior of the Gamma function see [23, (5.11)]). As a result, the propagator breaks down to a strictly decreasing exponential function, which tends to zero faster than any power law if we take the limit of spatial infinity, i.e. $m^2 \to \infty$. Since Δ and m^2 are related, the limit $R \to \infty$ corresponds to the spatial infinity. Thus we have shown that the integral vanishes in the limit.

In the same way, the evaluation of I_2 can be done, yielding the result $I_2 = \frac{1}{2}G_\Delta$. Hence we conclude that

$$\lim_{R\to\infty} I = \frac{1}{2}G_\Delta + \frac{1}{2}G_\Delta = G_\Delta,$$

which completes the claim. □

A similar construction is known as the Källén-Lehmann decomposition

$$\langle\phi(X)\phi(X')\rangle_{\text{int}} = \int_0^\infty \mathrm{d}m^2\, \rho(m^2)\Delta_m^+(X-X') \tag{3.38}$$

where a two-point function of an interacting quantum field theory is expressed by the sum of free propagators Δ_m^+ and the Källen-Lehmann measure $\rho(m^2)$, i.e. a spectral density function which is assumed to be positive definite. Thus a generalized free field can have the same two-point function as an interacting field which necessarily covers a continuum if masses extending to ∞ [41, Chapter 6.1].

In order to regulate the bulk propagator, we make explicit use of the spectral analysis.
First and foremost, consider the method of the heat kernel regularization scheme

upon the bulk propagator, i.e., introduce the scalar heat kernel of the Laplacian from its canonical definition [50]

$$K_\Delta(\tau; X, X') \equiv \operatorname{Tr} \exp\left[-\tau\left(-\Box_X + \nu^2 - \frac{1}{4}\right)\right]. \tag{3.39}$$

In terms of the continuous spectrum $E_\nu = \nu^2 + \frac{1}{4}, \forall \nu \in \mathbb{R}_{\geq 0}$ of the Laplacian [13], we state the following expression for the heat kernel.

Theorem 3.4.2. *The heat kernel formulation provides the homogeneous solution [34], [4]*

$$K_\Delta(\tau; X, X') = \frac{1}{L_{\text{AdS}}^2} \int_{-\infty}^{\infty} \mathrm{d}\nu \, \exp\left[-\frac{\nu^2 + (\Delta - \frac{1}{2})^2}{L_{\text{AdS}}^2}\tau\right] \Omega_\nu(X, X'), \tag{3.40}$$

on $\mathbb{H}^2$ satisfying the hyperbolic heat conduction equation

$$\left[\partial_\tau - \Box_X + \frac{\Delta(\Delta-1)}{L_{\text{AdS}}^2}\right] K_\Delta(\tau; X, X') = 0, \tag{3.41}$$

under the constraint of the boundary condition

$$\lim_{\tau \to 0} K_\Delta(\tau; X, X') = \delta(X, X'). \tag{3.42}$$

Proof. An explicit expression for the heat kernel can be found by applying the spectral analysis from before, i.e. we expand the heat kernel by eigenfunctions Ω_ν which are taken to be orthonormal with respect to the standard inner product on $\mathbb{H}^2$ satisfying (3.33). Hence, we assume for the heat kernel to have spectral representation

$$K_\Delta(\tau; X, X') = \int_{-\infty}^{\infty} \mathrm{d}\nu \, k_\Delta(\tau; \nu) \Omega_\nu(X, X').$$

We insert the expression into the hyperbolic heat conduction equation (3.41) :

$$\int_{-\infty}^{\infty} \mathrm{d}\nu \left[\frac{\partial k_\Delta(\tau;\nu)}{\partial \tau} \Omega_\nu(X, X') - k_\Delta(\tau;\nu) \Box_X \Omega_\nu(X, X') \right. \\ \left. + \frac{\Delta(\Delta-1)}{L_{\text{AdS}}^2} k_\Delta(\tau;\nu) \Omega_\nu(X, X')\right] = 0.$$

Using the eigenvalue equation (3.33), we obtain:

$$\int_{-\infty}^{\infty} \mathrm{d}\nu \left[\frac{\partial k_\Delta(\tau;\nu)}{\partial \tau} + \frac{\nu^2 + (\Delta - \frac{1}{2})^2}{L_{\text{AdS}}^2} k_\Delta(\tau;\nu)\right] \Omega_\nu(X, X') = 0.$$

The kernel then provides us a solvable differential equation

$$\left[\partial_\tau + \frac{\nu^2 + (\Delta - \frac{1}{2})^2}{L_{\text{AdS}}^2}\right] k_\Delta(\tau;\nu) = 0,$$

where the solution can be read off to be

$$k_\Delta(\tau;\nu) = C \exp\left[-\frac{\nu^2 + (\Delta - \frac{1}{2})^2}{L_{\text{AdS}}^2}\tau\right],$$

with C being a integration constant. One determines C by inserting the solution into the boundary condition of the heat kernel (3.42), while using the spectral representation of the heat kernel:

$$\begin{aligned}\lim_{\tau\to 0} K_\Delta(\tau; X, X') &= \lim_{\tau\to 0}\int_{-\infty}^{\infty} \mathrm{d}\nu\, C \exp\left[-\frac{\nu^2 + (\Delta - \frac{1}{2})^2}{L_{\mathrm{AdS}}^2}\tau\right]\Omega_\nu(X, X') \\ &= C\int_{-\infty}^{\infty} \mathrm{d}\nu\, \Omega_\nu(X, X') \\ &= C \cdot L_{\mathrm{AdS}}^2 \delta(X, X').\end{aligned}$$

In the last step, one uses the normalization condition of the harmonic functions (3.34), therefore giving us $C = L_{\mathrm{AdS}}^{-2}$ for the boundary condition to be fulfilled.

Hence, the heat kernel can written out to take the form

$$K_\Delta(\tau; X, X') = \frac{1}{L_{\mathrm{AdS}}^2}\int_{-\infty}^{\infty} \mathrm{d}\nu\, \exp\left[-\frac{\nu^2 + (\Delta - \frac{1}{2})^2}{L_{\mathrm{AdS}}^2}\tau\right]\Omega_\nu(X, X'),$$

which proves the claim. □

Note that the boundary condition (3.42) set in the theorem is important, since it ensures that the heat kernel behaves appropriately like the bulk propagator, which is obtained in the limit.

What is left is to show that the heat kernel K_Δ does represent the bulk propagator. By definition, the Euclidean AdS propagator of the Laplacian operator can be expressed by the heat kernel:

$$G_\Delta(X, X') = \int_0^\infty \mathrm{d}\tau\, K_\Delta(\tau; X, X'). \tag{3.43}$$

The integrable variable τ turns into a flow parameter, which provides a scale to regulate the bulk propagator. Thus we insert the expression (3.40) into (3.43) to obtain

$$\begin{aligned}G_\Delta(X, X') &= \frac{1}{L^2}\int_0^\infty \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\nu\, \exp\left[-\frac{\nu^2 + (\Delta - \frac{1}{2})^2}{L_{\mathrm{AdS}}^2}\tau\right]\Omega_\nu(X, X') \\ &= \frac{1}{L^2}\int_{-\infty}^{\infty} \mathrm{d}\nu\, \Omega_\nu(X, X')\int_0^\infty \mathrm{d}\tau\, \exp\left[-\frac{\nu^2 + (\Delta - \frac{1}{2})^2}{L_{\mathrm{AdS}}^2}\tau\right] \\ &= \int_{-\infty}^{\infty} \mathrm{d}\nu\, \frac{1}{\nu^2 + (\Delta - \frac{1}{2})^2}\Omega_\nu(X, X').\end{aligned}$$

As a result, it suffices to introduce cutoffs by replacing the borders along the flow parameter integration with the infrared cutoff Λ and ultraviolet cutoff Λ_0. Hence, we end up last but not least with a final expression for a regulated bulk propagator.

Corollary 1 (**Regulated propagator**). *The bulk propagator can be regulated in terms of an IR cutoff Λ and UV cutoff Λ_0 via the expression*

$$G_\Delta^{\Lambda,\Lambda_0}(X, X') = \int_{-\infty}^{\infty} \mathrm{d}\nu\, \frac{1}{\nu^2 + \left(\Delta - \frac{1}{2}\right)^2}\left[e^{-\frac{\nu^2 + \left(\Delta - \frac{1}{2}\right)^2}{L_{\mathrm{AdS}}^2 \Lambda_0^2}} - e^{-\frac{\nu^2 + \left(\Delta - \frac{1}{2}\right)^2}{L_{\mathrm{AdS}}^2 \Lambda^2}}\right]\Omega_\nu(X, X'). \tag{3.44}$$

Proof. To obtain a regulated bulk propagator, we make use again of the representation (3.43). Additionally introduce the restrictions upon on the integration of τ from above by an IR cutoff Λ and from below by an UV cutoff Λ_0 and use the explicit expression of the heat kernel (3.40)

$$
\begin{aligned}
G_\Delta^{\Lambda,\Lambda_0}(X,X') &= \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} \mathrm{d}\tau\, K_\Delta(\tau;X,X') \\
&= L_{\mathrm{AdS}}^{-2} \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\nu\, e^{-\left[\nu^2+\left(\Delta-\frac{1}{2}\right)^2\right] L_{\mathrm{AdS}}^{-2}\tau}\, \Omega_\nu(X,X') \\
&= L_{\mathrm{AdS}}^{-2} \int_{-\infty}^{\infty} \mathrm{d}\nu\, \Omega_\nu(X,X') \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} \mathrm{d}\tau\, e^{-\left[\nu^2+\left(\Delta-\frac{1}{2}\right)^2\right] L_{\mathrm{AdS}}^{-2}\tau} \\
&= -\int_{-\infty}^{\infty} \mathrm{d}\nu\, \frac{1}{\nu^2+\left(\Delta-\frac{1}{2}\right)^2} e^{-\left[\nu^2+\left(\Delta-\frac{1}{2}\right)^2\right] L_{\mathrm{AdS}}^{-2}\tau}\Bigg|_{\Lambda_0^{-2}}^{\Lambda^{-2}} \Omega_\nu(X,X') \\
&= \int_{-\infty}^{\infty} \mathrm{d}\nu\, \frac{1}{\nu^2+\left(\Delta-\frac{1}{2}\right)^2} \left[e^{-\frac{\nu^2+\left(\Delta-\frac{1}{2}\right)^2}{L_{\mathrm{AdS}}^2\Lambda_0^2}} - e^{-\frac{\nu^2+\left(\Delta-\frac{1}{2}\right)^2}{L_{\mathrm{AdS}}^2\Lambda^2}} \right] \Omega_\nu(X,X').
\end{aligned}
$$

□

We see that the regulated bulk propagator is positive definite, i.e.

$$
G_\Delta^{\Lambda,\Lambda_0}(X,X') \geq 0 \quad \forall \tau \in \mathbb{R}, \tag{3.45}
$$

since by definition $\Lambda \leq \Lambda_0$, which is important as the bulk propagator or two-point function is required to be positive in any case.

As the regulated bulk propagator is defined to be invariant under the isometries of AdS, the underlying cutoff regularization is therefore AdS invariant as well. Hence, correlation functions in the boundary theory are conformally invariant for all values of the cutoffs, i.e., they do not translate one-to-one between AdS and CFT but rather create a family of CFT's depending on the AdS space cutoffs.

Important to note is that, applied to a field with mass above the Breitenlohner-Freedman bound, the IR cutoff Λ does not regularize the theory and has more likely a technical use for instance to derive flow equations [30, Section 2.2]. The reason is AdS space acts as a symmetry-preserving IR regulator [15]. Hence, the only IR divergences to encounter are by evaluating conformal integrals at the boundary which contain those singularities by construction [35].

Now with the formulation of a cutoff regularization that can be translated to the boundary CFT, we bridge to an interacting scalar field theory in AdS space which we like to restrict to the boundary again to ultimately obtain an interacting CFT.

Chapter 4

Perturbation Theory

Quantum field theory is by definition not well-defined, as it considers an infinite collection of harmonic oscillators, thus giving infinite energy contribution. A possible solution is to redefine the Hamiltonian and subtract the contributing ground state [55] but mathematical objects, for instance correlation functions which contain information of the scattering amplitudes of a theory, hence describing interactions between asymptotic on-shell states via the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula remain ill-defined as they are naturally defined in the path integral formulation of QFT. Another example are quantum fields φ, which are usually decomposed into a series of modes. Their contributions at large $|\vec{k}|$ do not vanish rapidly enough in order to converge. Thus it is convenient to average the sums over the space region [53]. To be precise, quantum fields appear as operator valued distributions on the real Schwartz space $\mathscr{F}_{\mathbb{R}}$, where elements of its topological dual space $\mathscr{F}'_{\mathbb{R}}$ are called functionals Φ, treated as tempered distributions acting on a space of smooth, compactly supported test functions f [28]. Hence, we regard to functionals the scalar product of two Schwartz functions

$$\Phi(f) \equiv (\Phi, f) = \int_{\mathbb{R}^d} \mathrm{d}^d\vec{x}\, \varphi(\vec{x}) f(\vec{x}), \tag{4.1}$$

where ϕ is a function of an infinite number of variables. Thus, one approaches the problem that the measure of a functional

$$\mathcal{D}\varphi \equiv \prod_i \mathrm{d}\varphi(\vec{x}_i), \tag{4.2}$$

involves an infinite product of Lebesgue measures which does not exist. Consequently, a mathematical definition for correlation functions becomes difficult to formulate as functionals Φ are of infinite dimensions, thus ill-defined.

For a convergent perturbation theory, a functional integral can be expressed in a Euclidean quantum field theory which provides a much simpler structure, especially for correlation functions or Schwinger functions as there is no notion of time ordering or light cone singularities[1]. Also they satisfy the Wightman

[1] Except at coinciding points.

axioms while Wick rotating the corresponding Wightman distributions, proven by Osterwalder and Schrader. The underlying mathematical framework is given by the Gaussian integral formalism which free fields are described by a Gaussian measure, i.e. Euclidean fields are treated as random variables instead of operators on a Hilbert space. A more detailed introduction to this procedure is given in [48, Chapter 1.1].

In d-dimensional Euclidean quantum field theories, Schwinger functions are fully determined by their one- and two-point functions. In order to construct them, we consider a real scalar field φ and action $S_E[\varphi]$ which describe the partition function of Euclidean quantum field theories in the path integral formalism as the functional[2] [30, Section 3.1]

$$\mathcal{Z}_E[J] \equiv \mathcal{N} \int \mathcal{D}\varphi \, \exp\left[-S_E[\varphi] + \int \mathrm{d}^d\vec{x}\, J^A(\vec{x})\varphi_A(\vec{x}) \right], \tag{4.4}$$

where $J(\vec{x})$ is a fictitious source current, the integration measure $\mathcal{D}\varphi$ is a formal sum over all classical configurations of the field $\varphi(x)$ and $\mathcal{N}$ denotes a normalization constant which is chosen such that $Z[0] = 1$, hence

$$\mathcal{N} = \left(\int \mathcal{D}\varphi \, \exp\left(-S_E[\varphi]\right) \right)^{-1}. \tag{4.5}$$

The partition functional acts perfectly as generating functional for arbitrary n-point Euclidean correlation functions

$$\left\langle \prod_i^n \varphi(\vec{x}_i) \right\rangle_0 := \frac{1}{\mathcal{Z}_E[0]} \int \mathcal{D}\varphi \prod_i^n \varphi(\vec{x}_i) e^{-S_E(\varphi)}, \tag{4.6}$$

formally written as a functional average. They can be evaluated using partition functional (4.4) and functional derivatives with respect to the sources

$$\left\langle \prod_i^n \varphi(\vec{x}_i) \right\rangle_0 = \frac{1}{\mathcal{Z}_E[J]} \frac{\delta^n \mathcal{Z}_E[J]}{\prod_i^n \delta J(\vec{x}_i)} \bigg|_{J=0}. \tag{4.7}$$

The n-point Euclidean correlation functions is evaluated as the sum of all Feynman diagrams excluding vacuum bubbles. Thus, while not including any vacuum bubbles, the sum does include disconnected diagrams, which are diagrams where at least one external leg is not connected to all other external legs through some connected path. Excluding these disconnected diagrams instead defines connected n-point Euclidean correlation functions, generated by the functional

$$\mathcal{W}_E[J] = \ln \mathcal{Z}_E[J] \tag{4.8}$$

Note that connected is defined in the sense of the cluster decomposition, i.e., correlation functions vanish at large spacelike separations. For the case of two-point functions, we identify the two-point function as the bulk propagator G_Δ (3.26).

[2]Note that $\mathcal{Z}_E$ is exponentially damped. The action of the free theory is written in the form

$$S_E[\varphi] = \int \mathcal{D}\varphi \, \varphi(\Delta - m^2)\varphi, \tag{4.3}$$

where $\mathcal{D}\varphi$ denotes the Lebesgue measure on $\mathbb{R}^d$.

The goal in Euclidean quantum field theory for free scalar fields is to formulate those generating functionals in terms of a normalized Gaussian measure $\mathrm{d}\mu_C(\varphi)$ characterized by the positive operator

$$C(\vec{x}, \vec{x}') = \left(\Delta - m^2\right)^{-1}(\vec{x}, \vec{x}'). \tag{4.9}$$

For example in two dimensions, constructing the complete measure $Z_E^{-1}\exp\left(-S_E(\varphi)\right)\mathrm{d}\mu_C(\varphi)$ would fail to establish well-defined Wightman axioms due to divergences, for instance at coinciding points. Hence we require regularizations such that all regularized Schwinger functions exist. In Cor. 1, we were able to define a regularization scheme by the use of UV- and IR-cutoffs. Thus, we are going to review the mathematical setup in order to regard the regulated bulk propagator as the positive operator in (4.9).

Definition 4.0.1 (see [27, Definition 2.1]). *Let S_R be a sphere in $\mathbb{R}^d$ of radius $R \leq \infty$ centered at the origin. Let $V(\vec{x}, \vec{x}')$ be the kernel of a positive operator V on the set of square-integrable functions $L^2(S_R, \mathrm{d}^d x)$ and assume it to be continuous in $\vec{x}$ and $\vec{x}'$ such that*

$$\sup_{\vec{x}, \vec{x}' \in S_R} |V(\vec{x}, \vec{x}')| \leq K. \tag{4.10}$$

A functional Φ_V on

$$\mathscr{F} = \begin{cases} C^\infty_{0,\mathbb{R}}(S_R) & for\, R < \infty \\ \mathscr{F}_\mathbb{R}(\mathbb{R}^d) & for\, R = \infty \end{cases}, \tag{4.11}$$

is defined by

$$\Phi_V(f) = \exp\left(-\frac{1}{2}(f, Vf)\right), \tag{4.12}$$

which is normalized, i.e. $\Phi_V(0) = 1$ and continuous on $\mathscr{F}$.

For $d = 2$ and $R \to \infty$, we immediately recover

$$V(\vec{x}, \vec{x}') = \int \mathrm{d}^2\vec{x}\, \mathrm{d}^2\vec{x}' f_\kappa(\vec{x} - \vec{x}')C(\vec{x}', \vec{y}')f_\kappa(\vec{y} - \vec{y}'), \tag{4.13}$$

where $C(\vec{x}, \vec{x}')$ is the kernel of $(\Delta - m^2)^{-1}$ and f_κ is a positive, continuous function with the properties

$$\begin{aligned} &\mathrm{supp} f_\kappa \subseteq \left\{\vec{x} \in \mathbb{R}^d \,|\, |\vec{x}| \leq \frac{1}{\kappa}\right\}, \\ &f_\kappa(\vec{x}) = f_\kappa(-\vec{x}) \\ &\int \mathrm{d}^d\vec{x} = 1 \\ &f_\kappa(\vec{x}) \overset{\kappa\to\infty}{\longrightarrow} \delta(\vec{x}). \end{aligned} \tag{4.14}$$

The regulated bulk propagator $G_\Delta^{\Lambda,\Lambda_0}$ defined in (3.44) was constructed in such a way that it is positive definite and symmetric. As the definition requires, the covariance is a positive, continuous, non-degenerate bilinear form on the space of test functions [55, Section 3.1.1]. Therefore we identify the kernel of the operator $C(\vec{x}, \vec{x}')$ as the regulated Euclidean propagator of a free scalar field.

Regarding quantum field theory, predictions come from expectation values, one identifies them now with the moments of the measure. For Gaussian measures, they are rigorously defined due to Minlos' theorem.

Definition 4.0.2 (**Centred Gaussian integral (see [31, Chapter 2.1])**). *Consider a finite dimensional centered and normalized Gaussian measure defined by its covariance which is a symmetric, positive definite operator* $C^{\Lambda,\Lambda_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$*, depending on a UV-cutoff* Λ_0 *and IR-cutoff* Λ

$$\mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\varphi) = \mathcal{N} \prod_{\vec{x}} \exp\left(-S_E[\varphi]\right) \mathcal{D}\varphi(x), \tag{4.15}$$

where $\mathcal{N}$ *denotes an infinite normalization factor. Consequently its moments read*[3]

$$\int \mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\varphi) \prod_i^n \varphi(\vec{x}_i) = \sum_\pi \prod_{l\in\pi} C^{\Lambda,\Lambda_0}(\vec{x}_i, \vec{x}_j), \tag{4.16}$$

where the sum runs over all partitions π *of the set* $\{1, \dots, n\}$ *into unordered pairs* $l = (i, j)$*. The measure is normalized such that*

$$\int \mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\varphi) = 1. \tag{4.17}$$

For finite cutoffs, the covariance is assumed to be a smooth function, thus the Gaussian measure is supported on smooth functions $\varphi \in \mathscr{F}$. As one removes the cutoffs, i.e. taking the physical limit $\Lambda \to 0$ (thermodynamic limit) and $\Lambda_0 \to \infty$ (continuum limit), the covariance turns into the free bulk propagator G_Δ and the measure is supported on a distributional context [31].
Note that the defined normalization (4.17) leads to a general characteristic functional [48, Section 1.2.2]

$$\int_{\mathscr{F}'} \mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\Phi) \exp\left(i\Phi(f)\right) = \exp\left(-\frac{1}{2} C^{\Lambda,\Lambda_0}(f,f)\right). \tag{4.18}$$

The left-hand side of (4.18) is known as the generating function of the Gaussian measure. Expanding both sides expresses the moments of the measure as polynomials in the covariance like in (4.16). Thus one can assign an average with respect to the centered Gaussian measure $\mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\Phi)$ to obtain regulated expectation values in the free theory by Gaussian integrals

$$\left\langle \prod_{i=1}^n \Phi_i(f) \right\rangle_0^{\Lambda,\Lambda_0} \equiv \int_{\mathscr{F}'} \mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\Phi) \prod_{i=1}^n \Phi_i(f), \tag{4.19}$$

where Φ_i is a $\mathrm{d}\mu_C^{\Lambda,\Lambda_0}$-integrable local functional on $\mathscr{F}'$ [27]. By using Def. 4.0.2 and inserting the identities into the generating functional for Euclidean correlation functions (4.4), we canonically construct interacting expectation values in a regulated context, which is often regarded by the use of the Euclidean

[3] The right-hand side vanishes if n is odd.

Gell-Man-Low formula [31]

$$\begin{aligned}\left\langle \prod_{i=1}^{n} \Phi_i(f) \right\rangle_{\text{int}}^{\Lambda,\Lambda_0} &= \frac{\int \mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\Phi) \prod_{i=1}^{n} \Phi_i(f) \exp\left(-S_{\text{int}}[\Phi]\right)}{\int \mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\Phi) \exp\left(-S_{\text{int}}[\Phi]\right)} \\ &= \frac{\langle \prod_{i=1}^{n} \Phi_i(f) \exp\left(-S_{\text{int}}[\Phi]\right)\rangle_0^{\Lambda,\Lambda_0}}{\langle \exp\left(-S_{\text{int}}[\Phi]\right)\rangle_0^{\Lambda,\Lambda_0}},\end{aligned} \tag{4.20}$$

where S_{int} is the interaction term which is assumed to be bounded from below. Several identities and formulations can be identified in the same way in [58] for some deeper explanations about the connections of Gaussian measures to the Euclidean quantum field theory.

4.1 Sine-Gordon Theory

We begin the perturbation theory by first choosing an interaction model, which will be the Sine-Gordon model. It is a well-studied example of a two-dimensional interacting quantum field theory with a well-known phase structure and even an renormalization group flow that is not considered to fit in the general scheme [44]. It is formulated with a nonlinear interaction term that gives rise to soliton and breather solutions, which can be interpreted as geodesics in PAdS^2. Specifically, soliton solutions correspond to closed geodesics, while breather solutions correspond to open geodesics that extend to infinity.

Its Euclidean action expands to the form [31]

$$S[\phi] = \int_{\text{PAdS}^2} \mathrm{d}^2\vec{x}\, \sqrt{g} \left[\frac{1}{2}\partial^\mu \phi(\vec{x})\partial_\mu \phi(\vec{x}) + \frac{1}{2} m^2 \phi(\vec{x})^2 - 2\lambda(\vec{x}) \cos\left(\beta\phi(\vec{x})\right)\right], \tag{4.21}$$

where $m \geq 0$ is a mass parameter, $\beta > 0$ is a coupling constant and λ is an interaction cutoff which is assumed to be a function of compact support on $\mathscr{F}$, i.e. the interaction is concentrated on a bounded, open set $V \in \text{PAdS}^2$, making perturbative expansion possible to perform [8, Section 1]. If $\lambda(\vec{x}) = 0$, the Sine-Gordon model degenerates to a continuum Gaussian free field with covariance $(-\Delta + m^2)^{-1}$ [6]. For the purpose of a quantum approach, no references to the classical theory of Sine-Gordon will be done.

First, some results of the investigation of such quantum theory will be given. In the perspective of two-dimensional scalar field theory with non-derivative interactions, all UV correlated divergences that might appear in any order of perturbation theory are related to closed loops consisting of a single internal line, i.e. graphs in which two fields at the same vertex are contracted with each other [18, Section 1&2]. Depending on the dimensionless coupling β which is not renormalized, the quantum Sine-Gordon is divided into an ionized phase corresponding to massless, strongly coupled and non-renormalizable theories for $\beta^2 > 8\pi$ and a molecular phase representing massive, weakly coupled and renormalizable theories for $\beta^2 < 8\pi$ [44, Section 1]. In the molecular phase, all the UV divergences can be removed by normal-ordering the Hamiltonian [28] which is displayed as a multiplicative renormalization of the dimensional coupling m^2. Thus, no wave-function renormalization for the elementary field is necessary [51, Section 3.2]. We display the regularization of the Sine-Gordon model by first bounding the space in the interaction to be finite, as mentioned in the beginning of the section, and second normal-ordering the kernel of the interaction term [8]

$$S_{\mathrm{int}}[\varphi] = -2 \int_V \mathrm{d}^2\vec{x}\, \sqrt{g}\lambda(\vec{x}) :\cos\left(\beta\varphi(\vec{x})\right):_C . \tag{4.22}$$

An important part of perturbation theory is to define a particular normal-ordering scheme which is responsible to subtract the infinities in the vacuum expectation values of products of fields.

Definition 4.1.1 (**Normal-ordering (see [55])**). *A normal-ordering operation* $:\cdot:_C$ *with respect to a given covariance* C *is defined recursively by*

$$\begin{aligned} :\Phi(f)^0:_C &= 1 \\ \frac{\delta}{\delta\Phi} :\Phi(f)^n:_C &= n :\Phi(f)^{n-1}:_C \qquad & n = 1,2,\dots \\ \int \mathrm{d}\mu_C^{\Lambda,\Lambda_0}(\Phi)\, :\Phi(f)^n:_C &= 0 & n = 1,2,\dots \end{aligned} \tag{4.23}$$

and acts as a linear operation.

In [29, Theorem 2.1] & [28], Fröhlich and Seiler proved for sufficiently small $\left|\lambda/m^2\right|$ the convergence of the Feynman perturbative series in λ for Euclidean correlation functions of a massive Sine-Gordon theory in an infinite volume for $\beta^2 < 4\pi$. Additionally, it is stated that in the setup for $\lambda \in \mathbb{R}$, Euclidean correlation functions uniquely determine a relativistic quantum field theory satisfying all Wightman axioms. For a convergent perturbation theory in the bulk, the interaction term can be interpret in terms of a sum of vertex operators expressed by an exponential function of the field arising by the identity

$$V_\beta(\vec{x}) + V_{-\beta}(\vec{x}) :=: \exp\left(i\beta\varphi(\vec{x})\right):_C + :\exp\left(-i\beta\varphi(\vec{x})\right):_C = 2 :\cos\left(\beta\varphi(\vec{x})\right):_C . \tag{4.24}$$

One then extends the definition of the normal-ordering to vertex operators. Therefore the Ansatz

$$V_{\pm\beta}(\vec{x}) = N(\Lambda,\Lambda_0)\exp\left(\pm i\beta\varphi(\vec{x})\right), \tag{4.25}$$

is chosen carefully since the normal-ordering operation is formulated with respect to its covariance C which again is equivalent to the Euclidean AdS propagator, regulated by the cutoffs. Hence it is expected within the Gaussian formalism to extract a normalization parameter N continuously dependable on the cutoffs. To find the expression of such parameter, one interprets the vertex operator to be a random variable, allowing to use the condition set by Gaussian expectation values for exponentials [52]

$$\langle V_{\pm\beta}(\vec{x})\rangle_0^{\Lambda,\Lambda_0} = 1. \tag{4.26}$$

Plugging the Ansatz into the condition (4.26) and expand the exponential into its power series yields

$$1 \stackrel{!}{=} N(\Lambda,\Lambda_0)\left\langle\exp\left(\pm i\beta\varphi(\vec{x})\right)\right\rangle_0 = N(\Lambda,\Lambda_0)\sum_{k=0}^{\infty}\frac{(i\beta)^k}{k!}\left\langle\varphi(\vec{x})^k\right\rangle_0 \tag{4.27}$$

The expression $\langle \varphi(\vec{x})^k \rangle_0$ is dealt by Wicks theorem which decomposes the n-point function into polynomials of two-point functions of fields $\varphi(\vec{x}_i)$. Since we are dealing with products of fields at the same point, one need to take the coinciding point limit at the end of the calculation. For that reason, one is allowed to sum up the pairs to find a sequence which suits the power series of an exponential function again, therefore obtaining

$$\begin{aligned} 1 &\overset{!}{=} N(\Lambda, \Lambda_0) \sum_{k=0}^{\infty} \frac{(2k-1)!(i\beta)^{2k} C^{\Lambda,\Lambda_0}(\vec{x},\vec{x})^k}{2^{k-1} 2k!(k-1)!} \\ &= N(\Lambda, \Lambda_0) \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{1}{2} \beta^2 C^{\Lambda,\Lambda_0}(\vec{x},\vec{x}) \right)^k \\ &= N(\Lambda, \Lambda_0) \exp\left(-\frac{1}{2} \beta^2 C^{\Lambda,\Lambda_0}(\vec{x},\vec{x}) \right). \end{aligned} \tag{4.28}$$

In summary, one inserts the result into the Ansatz (4.25) to find [31]

$$V_{\pm\beta}(\vec{x}) = \exp\left(\frac{1}{2} \beta^2 C^{\Lambda,\Lambda_0}(\vec{x},\vec{x}) \pm i\beta\varphi(\vec{x}) \right). \tag{4.29}$$

The representation of the vertex operator in a functional form can be recovered in an equivalent way by using the notation of Wick exponentials [55]

$$: \exp\left(i\Phi(f)\right) :_C \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} : \Phi(f)^n :_C = \exp\left(\frac{1}{2} G_{\Delta}^{\Lambda,\Lambda_0}(f,f) \right), \tag{4.30}$$

defined as a formal power series in the spacetime function $f : \mathrm{AdS}^2 \to \mathbb{C}$, and additionally considering the properties defined in Def. 4.1.1 to find [18]

$$V_{\pm\beta}(\Phi(f)) = \exp\left(\frac{1}{2} C^{\Lambda,\Lambda_0}(f,f) + i\Phi(f) \right). \tag{4.31}$$

The field φ here is smeared with the test function $f(\vec{x}) = \pm\beta\delta^2(\vec{x}-\vec{x}')$. Comparing both equations (4.29) and (4.31), one finds the contribution of the covariance at the coinciding point limit, i.e., $\vec{x} = \vec{x}'$. Clearly, as the covariance is equivalent to the regulated Euclidean AdS propagator, it diverges as one approaches the physical limit $\Lambda \to 0$, $\Lambda_0 \to \infty$ in the same way.

To avoid the divergences produced by the contributions of the coinciding point propagator, one extends the expectation value to n vertex operators, leaving us with calculations

$$\begin{aligned} \left\langle \prod_{i=1}^{n} V_{\beta_i}(\vec{x}_i) \right\rangle_0^{\Lambda,\Lambda_0} &= \left\langle \prod_{i=1}^{n} : \exp\left(i\beta_i\varphi(\vec{x}_i)\right) :_C \right\rangle_0^{\Lambda,\Lambda_0} \\ &= \exp\left(\frac{1}{2} \sum_{i=1}^{n} \beta_i^2 C^{\Lambda,\Lambda_0}(\vec{x}_i,\vec{x}_i) + i \sum_{i=1}^{n} \beta_i \varphi(\vec{x}_i) \right) \\ &= \exp\left(- \sum_{1\leq i<j\leq n} \beta_i\beta_j C^{\Lambda,\Lambda_0}(\vec{x}_i,\vec{x}_j) \right). \end{aligned} \tag{4.32}$$

Here we used the fact that the product of the vertex operators can also be written in a functional form, like in [27, Lemma 2.2] by the promotion of $\sum_{i=1}^n \beta_i \varphi(\vec{x}_i)$ to

$$\Phi(f) \equiv (f, \Phi) = \int f(\vec{x})\varphi(\vec{x})\,\mathrm{d}^2\vec{x}, \tag{4.33}$$

with $f(\vec{x}) = \sum_{i=1}^n \beta_i \delta^2(\vec{x} - \vec{x}_i)$ and additionally expressing regulated bulk propagator in its functional form

$$G_{\Delta}^{\Lambda,\Lambda_0}(f,g) = \int\int \mathrm{d}^2\vec{x}\,\mathrm{d}^2\vec{x}'\, G_{\Delta}^{\Lambda,\Lambda_0}(\vec{x},\vec{x}')f(\vec{x})g(\vec{x}'). \tag{4.34}$$

The result in (4.32) has also been found in [28, Lemma 4.1], which was proven through the characteristic functional and the corresponding normal-ordering operation of exponentials.

In conclusion, one finds the n-point correlation functions of vertex operators not contributing to any propagators yielding any UV divergences which is a common result in two-dimensional quantum field theories performing normal-ordering upon the Hamiltonian. Hence, one is allowed to take the physical limit to replace the regulated covariance with the free covariance

$$\left\langle \prod_{i=1}^n V_{\beta_i}(\vec{x}_i) \right\rangle_0^{0,\infty} = \exp\left(- \sum_{1\leq i<j\leq n} \beta_i\beta_j C(\vec{x}_i,\vec{x}_j) \right). \tag{4.35}$$

This result is emphasized by the convergence of the cluster expansion leading to the transfer of analytical properties of finite volume Schwinger functions to infinite volume objects [29]. In the following the superscript of the physical limit is omitted due to notation clustering, thus one always considers this limit.

4.2 Two-point function

As we have managed to prescribe an expression for the n-point correlation function of normal-ordered vertex operators with any contributing diagram, we like to restrict this by considering only the connected correlation functions of normal-ordered vertex operators. Thus, we need to construct the formulas again in order to perform the calculation.
Luckily, we already discussed a generating functional for connected correlation functions in (4.7), which is given by

$$\left\langle \prod_{i}^n V_{\beta_i}(\vec{x}_i) \right\rangle_c = \left. \frac{\delta^n \ln \left\langle \exp\left(J^A(\vec{x})V_{\beta_A}(\vec{x}_A)\right)\right\rangle_0}{\prod_i^n \delta J(\vec{x}_i)} \right|_{J=0}, \tag{4.36}$$

where we exchanged the field with the normal-ordered vertex operator and used the expression of the regulated expectation values in (4.19) while performing the physical limit, which we justified in the previous section.

The connected two-point correlation function for normal-ordered vertex operators is given for $n = 2$ in (4.36)

$$\begin{aligned}\langle V_{\beta_1}(\vec{x}_1)V_{\beta_2}(\vec{x}_2)\rangle_c &= \left.\frac{\delta^2 \ln \langle \exp\left(J_1(\vec{x}_1)V_{\beta_1}(\vec{x}_1) + J_2(\vec{x}_2)V_{\beta_2}(\vec{x}_2)\right)\rangle_0}{\delta J_1(\vec{x}_1)\delta J_2(\vec{x}_2)}\right|_{J_1=J_2=0} \\ &= \langle V_{\beta_1}(\vec{x}_1)V_{\beta_2}(\vec{x}_2)\rangle_0 - \langle V_{\beta_1}(\vec{x}_1)\rangle_0 \langle V_{\beta_2}(\vec{x}_2)\rangle_0 \\ &= \exp\left(-\beta_1\beta_2 G_\Delta(\vec{x}_1,\vec{x}_2)\right) - 1,\end{aligned} \tag{4.37}$$

where we used the n-point correlation function of normal-ordered vertex operators (4.35) and the normalization condition (4.26).

The main task of the thesis is to formulate a dictionary between the fields in the bulk, displayed by the vertex operators, and the operators on the boundary of the bulk space. To achieve the duality, one sends the vertex operators towards the boundary of AdS, which we remember, is achieved by taking the limit $z_i \to 0$ of the corresponding radial coordinates. Thus we find the bulk propagator, appearing in (4.37) to develop under Weyl-rescaling to the boundary propagator

$$G_\Delta(\vec{x}_1,\vec{x}_2) \xrightarrow{[z_1,z_2\to\partial\text{AdS}]} z_1^\Delta z_2^\Delta c_\Delta \left|p_1 - p_2\right|^{-2\Delta}. \tag{4.38}$$

Since G_Δ is even infinitely differentiable at the evaluation point by the fact that the invariant distance $u(\vec{x},\vec{x}')$ is blowing up while approaching the boundary, thus leading to a vanishing bulk propagator and ultimately allowing us to expand the exponential in (4.37) to $1^{\underline{\text{st}}}$ leading order of the Maclaurin series

$$\exp\left(-\beta_1\beta_2 G_\Delta(\vec{x}_1,\vec{x}_2)\right) = \sum_{k=0}^{\infty} \frac{\left(-\beta_1\beta_2 G_\Delta(\vec{x}_1,\vec{x}_2)\right)^k}{k!}. \tag{4.39}$$

Consequently, this discussion concludes the connected two-point correlation function of normal-ordered vertex operators in the free theory

$$\langle V_{\beta_1}(\vec{x}_1)V_{\beta_2}(\vec{x}_2)\rangle_c = (i\beta_1)(i\beta_2) z_1^\Delta z_2^\Delta c_\Delta \left|p_1 - p_2\right|^{-2\Delta}. \tag{4.40}$$

We observe a similar form of the two-point correlation function of operators acting on the boundary, like in (2.38), which we define here to be

$$\langle \mathcal{O}_\Delta(p_1)\mathcal{O}_\Delta(p_2)\rangle^{\text{BCFT}} = c_\Delta \left|p_1 - p_2\right|^{-2\Delta}. \tag{4.41}$$

One then identifies a direct correspondence to the connected two-point correlation function of normal-ordered vertex operators sent to the boundary, which decodes the formulation of a dictionary between the operators on the boundary CFT dual to the bulk fields in the AdS space

$$\mathcal{O}_\Delta(p) = \frac{1}{(i\beta)} \lim_{z\to 0} z^{-\Delta} V_\beta(\vec{x}). \tag{4.42}$$

This dictionary represents a general result of the underlying free theory in any free quantum field theory dual to a boundary conformal field theory. Here the vertex operator gets re-scaled in the same way as we did in the AdS/CFT correspondence when we sent the bulk propagator to the boundary.

Finally, one is able to extend the previous calculation to an interacting theory. Adding self-interacting effects requires an additional term S_{int} in the action, which is set by the sine-Gordon model and expressed in terms of vertex operators

$$S_{\text{int}} = -\lambda \int_{\text{PAdS}_2} \mathrm{d}^2\vec{x}_\xi \, \sqrt{g} \left[V_\xi(\vec{x}_\xi) + V_{-\xi}(\vec{x}_\xi) - 2 \right], \tag{4.43}$$

assuming λ to be small. Here one chooses to subtract by a constant piece in order to cancel divergences such as the vacuum graphs in future calculations [2, Section 3.2.2]. Also note that the vertex operators arising from the self-interacting part are taken to be fixed in the bulk.

This formulation suits the previous mathematical groundwork for correlation functions to compute the self-interacting two-point correlation function up to $1^{\underline{\text{st}}}$ order in λ[4]

$$\begin{aligned}
\langle V_{\beta_1}(\vec{x}_1) V_{\beta_2}(\vec{x}_2) \rangle_c^{\text{int}} &= \frac{\delta^2}{\delta J_1(\vec{x}_1) \delta J_2(\vec{x}_2)} \\
&\quad \times \ln \frac{\langle \exp\left(J_1(\vec{x}_1) V_{\beta_1}(\vec{x}_1) + J_2(\vec{x}_2) V_{\beta_2}(\vec{x}_2) - S_{\text{int}} \right) \rangle_0}{\langle \exp\left(-S_{\text{int}}\right) \rangle_0} \Bigg|_{J_1 = J_2 = 0} \\
&= \frac{\langle V_{\beta_1}(\vec{x}_1) V_{\beta_2}(\vec{x}_2) \exp\left(-S_{\text{int}}\right) \rangle_0}{\langle \exp\left(-S_{\text{int}}\right) \rangle_0} \\
&\quad - \frac{\langle V_{\beta_1}(\vec{x}_1) \exp\left(-S_{\text{int}}\right) \rangle_0}{\langle \exp\left(-S_{\text{int}}\right) \rangle_0} \frac{\langle V_{\beta_2}(\vec{x}_2) \exp\left(-S_{\text{int}}\right) \rangle_0}{\langle \exp\left(-S_{\text{int}}\right) \rangle_0},
\end{aligned} \tag{4.44}$$

One expands the denominator up to $1^{\underline{\text{st}}}$ order, inserts (4.43) and uses the normalization condition (4.26) in the physical limit to obtain

$$\langle \exp\left(-S_{\text{int}}\right) \rangle_0 \approx 1. \tag{4.45}$$

Performing the same expansion for the nominator and pushing the operators V_{β_1}, V_{β_2} to the boundary, leads to the simplification [2, Section 3.2.2]

$$\begin{aligned}
\langle V_{\beta_1}(\vec{x}_1) V_{\beta_2}(\vec{x}_2) \rangle_c^{\text{int}} \approx z_1^\Delta z_2^\Delta (i\beta_1)(i\beta_2) &\left[c_\Delta \left| p_1 - p_2 \right|^{-2\Delta} - 2\lambda \xi^2 I_\Delta(p_1, p_2) \right] \\
&- z_1^{2\Delta} z_2^{2\Delta} (i\beta_1)^2 (i\beta_2)^2 c_\Delta \left| p_1 - p_2 \right|^{-2\Delta} 2\lambda \xi^2 I_\Delta(p_1, p_2)
\end{aligned} \tag{4.46}$$

where in $1^{\underline{\text{st}}}$ order perturbation I_Δ denotes the conformal integral

$$I_\Delta(p_1, p_2) = \int_{\text{PAdS}_2} \mathrm{d}^2\vec{x}_\xi \, \sqrt{g} G_{\partial\Delta}(\vec{x}_\xi, p_1) G_{\partial\Delta}(\vec{x}_\xi, p_2), \tag{4.47}$$

with result (see Appendix A)

$$\begin{aligned}
I_\Delta(p_1, p_2) = &\frac{L_{\text{AdS}}^2}{2\Delta - 1} c_\Delta \left| p_1 - p_2 \right|^{-2\Delta} \\
&\times \left[2\ln\left(\frac{M}{\epsilon} \left| p_1 - p_2 \right| \right) - \gamma - \psi_0(\Delta) + \psi_0\left(\Delta - \frac{1}{2} \right) \right].
\end{aligned} \tag{4.48}$$

[4]There is no use to compute λ to much higher orders since the perturbation series is supposed to have zero radius of convergence. Thus the $1^{\underline{\text{st}}}$ order terms of the asymptotic series give a reasonable approximation.

Firstly, it can be observed that a logarithmic IR divergence is coming from the internal AdS loop integral, which is a general observation of integrations over AdS space [24]. Hence, by introducing a regulator for the computation of the logarithmic infrared divergent integral which stops the integration at a distance ϵ away from the AdS boundary, the loop integral gives out a finite result, allowing us to perform the integral, where we take the limit $\epsilon \to 0$ afterwards. Plugging the Ansatz $\Delta_1 = \Delta + \lambda\Delta_{(1)}$ into (4.41) and performing a Taylor expansion, we are able to renormalize the scaling dimension of the dual operator corresponding to the first term two-point function in (4.46), thus giving the so-called anomalous dimension[5]

$$\Delta_1 = \Delta - \lambda\xi^2 L_{\text{AdS}}^2 \frac{2}{2\Delta - 1}. \tag{4.49}$$

Looking at the CFT point of view, the dictionary between the vertex and dual operator can define a renormalization factor Z which ultimately relates the renormalized operator $\mathcal{O}^{\text{ren}}$ to the bare operator $\mathcal{O}$. Thus the divergent cutoff ϵ has to be absorbed in Z. Therefore we find by the same Taylor expansion in λ to $1^{\underline{\text{st}}}$ order

$$Z(\epsilon) = 1 - \lambda\xi^2 L_{\text{AdS}}^2 \frac{2}{2\Delta - 1}\left(2\ln\left(\frac{M}{\epsilon}\right) - \gamma - \psi_0(\Delta) + \psi_0\left(\Delta - \frac{1}{2}\right) + C\right), \tag{4.50}$$

where a renormalization scale M has been introduced which is set to be arbitrary, hence it displays no restriction in practice [30]. We also added a counterterm denoted by C, which gives the opportunity to renormalize by multiplication the operator. It has to be set in a way, that the divergence in the expression (4.50) can be treated to a finite result in the limit $\epsilon \to 0$. Thus $C = 2\ln\epsilon$.

As a result, we obtain as a result the renormalized dictionary

$$\mathcal{O}_{\Delta_1}^{\text{ren}}(p) = \frac{1}{(i\beta)} \lim_{\epsilon\to 0} \sqrt{Z(\epsilon)} \lim_{z\to 0} z^{-\Delta_1} V_\beta(\vec{x})\big|_{z=\epsilon}. \tag{4.51}$$

The second term in (4.46) is a correction term that scales differently to the first term. Hence, one assumes that the vertex operator is split into the primary operator $\mathcal{O}$ and a logarithmic multiplet $\tilde{\mathcal{O}} := \{\hat{\mathcal{O}}, :\mathcal{O}^2:\}$ (for a detailed discussion see Section 2.2.3). Both operators of $\tilde{\mathcal{O}}$ are assumed to scale with 2Δ where $:\mathcal{O}^2 := \mathcal{O}_{2\Delta}$. Their renormalizations are as follows:

$$\begin{aligned} \tilde{\mathcal{O}}_1^{\text{ren}} &= \lim_{\epsilon\to 0} \sqrt{A(\epsilon)} :\mathcal{O}^2: \\ \tilde{\mathcal{O}}_2^{\text{ren}} &= \lim_{\epsilon\to 0} \sqrt{Z(\epsilon)}\hat{\mathcal{O}}. \end{aligned} \tag{4.52}$$

The first renormalized operator undergoes an additive renormalization with

$$A(\epsilon) := \frac{c_{\Delta_1}}{c_\Delta} - \lambda\xi^2 L_{\text{AdS}}^2 \frac{2}{2\Delta - 1}\left[2\ln\left(\frac{M}{\epsilon}\right)\right], \tag{4.53}$$

and the second performs the same multiplicative renormalization which was defined before. These renormalized values of parameters are then considered as physical ones, compared with experiments.

[5]The anomalous dimension is related to the corrected bulk mass by $\Delta_1(\Delta_1 - 1) = L_{\text{AdS}}^2 m^2$ [35].

4.3 Three-point correlation function

We employ the same considerations for the connected three-point correlation function of normal-ordered vertex operators. Hence, for $n = 3$ in (4.36) we obtain

$$\begin{aligned}\langle V_{\beta_1} V_{\beta_2} V_{\beta_3} \rangle_c = &\exp\left(-\beta_1\beta_2 G_\Delta(\vec{x}_1, \vec{x}_2) - \beta_1\beta_3 G_\Delta(\vec{x}_1, \vec{x}_3) - \beta_2\beta_3 G_\Delta(\vec{x}_2, \vec{x}_3)\right) \\ &- \exp\left(-\beta_1\beta_2 G_\Delta(\vec{x}_1, \vec{x}_2)\right) - \exp\left(-\beta_1\beta_3 G_\Delta(\vec{x}_1, \vec{x}_3)\right) \\ &- \exp\left(-\beta_2\beta_3 G_\Delta(\vec{x}_2, \vec{x}_3)\right) + 2.\end{aligned} \tag{4.54}$$

Again pushing the vertex operators towards the boundary and expanding all exponentials separately to $1^{\underline{\text{st}}}$ leading order, we end up with the connected three-point correlation function of normal-ordered vertex operators in the free theory

$$\begin{aligned}\langle V_{\beta_1} V_{\beta_2} V_{\beta_3} \rangle_c \approx &(i\beta_1)^2 (i\beta_2)^2 (i\beta_3)^2 z_1^{2\Delta} z_2^{2\Delta} z_3^{2\Delta} \langle \mathcal{O}_{2\Delta}(p_1) \mathcal{O}_{2\Delta}(p_2) \mathcal{O}_{2\Delta}(p_3) \rangle \\ &+ (i\beta_1)^2 (i\beta_2)(i\beta_3) z_1^{2\Delta} z_2^{\Delta} z_3^{\Delta} \langle \mathcal{O}_{2\Delta}(p_1) \mathcal{O}_{\Delta}(p_2) \mathcal{O}_{\Delta}(p_3) \rangle \\ &+ (i\beta_1)(i\beta_2)^2 (i\beta_3) z_1^{\Delta} z_2^{2\Delta} z_3^{\Delta} \langle \mathcal{O}_{\Delta}(p_1) \mathcal{O}_{2\Delta}(p_2) \mathcal{O}_{\Delta}(p_3) \rangle \\ &+ (i\beta_1)(i\beta_2)(i\beta_3)^2 z_1^{\Delta} z_2^{\Delta} z_3^{2\Delta} \langle \mathcal{O}_{\Delta}(p_1) \mathcal{O}_{\Delta}(p_2) \mathcal{O}_{2\Delta}(p_3) \rangle .\end{aligned} \tag{4.55}$$

Note that the dictionary defined in (4.42) has been used throughout the derivation, which presents the same observation for the three-point case.

As we did for the two-point correlation function, we add interactions and see if the given operators from the two-point correlation function encode a similar structure for the three-point functions. Hence we are able to express them into

$$\begin{aligned}\langle V_{\beta_1} V_{\beta_2} V_{\beta_3} \rangle_c^{\text{int}} \approx &(i\beta_1)^2 (i\beta_2)^2 (i\beta_3)^2 z_1^{2\Delta_1} z_2^{2\Delta_1} z_3^{2\Delta_1} \langle \mathcal{O}_{2\Delta_1}(p_1) \mathcal{O}_{2\Delta_1}(p_2) \mathcal{O}_{2\Delta_1}(p_3) \rangle^{\text{ren}} \\ &+ (i\beta_1)^2 (i\beta_2)(i\beta_3) z_1^{2\Delta_1} z_2^{\Delta} z_3^{\Delta} \langle \mathcal{O}_{2\Delta_1}(p_1) \mathcal{O}_{\Delta}(p_2) \mathcal{O}_{\Delta}(p_3) \rangle^{\text{ren}} \\ &+ (i\beta_1)(i\beta_2)^2 (i\beta_3) z_1^{\Delta} z_2^{2\Delta_1} z_3^{\Delta} \langle \mathcal{O}_{\Delta}(p_1) \mathcal{O}_{2\Delta_1}(p_2) \mathcal{O}_{\Delta}(p_3) \rangle^{\text{ren}} \\ &+ (i\beta_1)(i\beta_2)(i\beta_3)^2 z_1^{\Delta} z_2^{\Delta} z_3^{2\Delta_1} \langle \mathcal{O}_{\Delta}(p_1) \mathcal{O}_{\Delta}(p_2) \mathcal{O}_{2\Delta_1}(p_3) \rangle^{\text{ren}} \\ &+ (i\beta_1)^2 (i\beta_2)^2 (i\beta_3)^2 z_1^{2\Delta_1} z_2^{2\Delta_1} z_3^{2\Delta_1} \left\langle \tilde{\mathcal{O}}_{2\Delta_1}(p_1) \tilde{\mathcal{O}}_{2\Delta_1}(p_2) \tilde{\mathcal{O}}_{2\Delta_1}(p_3) \right\rangle^{\text{ren}} \\ &+ (i\beta_1)^2 (i\beta_2)(i\beta_3) z_1^{2\Delta_1} z_2^{\Delta} z_3^{\Delta} \left\langle \tilde{\mathcal{O}}_{2\Delta_1}(p_1) \mathcal{O}_{\Delta}(p_2) \mathcal{O}_{\Delta}(p_3) \right\rangle^{\text{ren}} \\ &+ (i\beta_1)(i\beta_2)^2 (i\beta_3) z_1^{\Delta} z_2^{2\Delta_1} z_3^{\Delta} \left\langle \mathcal{O}_{\Delta}(p_1) \tilde{\mathcal{O}}_{2\Delta_1}(p_2) \mathcal{O}_{\Delta}(p_3) \right\rangle^{\text{ren}} \\ &+ (i\beta_1)(i\beta_2)(i\beta_3)^2 z_1^{\Delta} z_2^{\Delta} z_3^{2\Delta_1} \left\langle \mathcal{O}_{\Delta}(p_1) \mathcal{O}_{\Delta}(p_2) \tilde{\mathcal{O}}_{2\Delta_1}(p_3) \right\rangle^{\text{ren}} .\end{aligned} \tag{4.56}$$

We see that the expression is split into two terms again. Each of the terms satisfies the dictionary of the two-point case in the same way as the three-point correlation function in the free theory (4.55), even for the logarithmic multiplet! Thus it is sufficient to conclude, that the dictionary is able to construct higher point correlation functions as well.

Chapter 5

Summary and Outlook

We summarize what have been done in the thesis. First, we made contact with the structure of the assumed correspondence which might lure between Anti-de Sitter spaces and CFT's. A key result was the connection between the fields in the bulk and a scalar operator on a quantum field theory located at its boundary. The dictionary which originates out of the correspondence, translates objects like correlation functions to the desired theory.
Thus it is natural to construct a quantum field theory of the bulk in a way that one is allowed to apply the dictionary in a well-defined matter. By the result of the free bulk propagator, we were able to proceed with perturbation theory as a necessary tool to handle divergences that appear while considering an interacting QFT. Thus we chose a Sine-Gordon interaction model in two dimensions to exploit the given vertex operator formalism, which encode a suitable way to define well-defined correlation functions, ultimately allowing us to formulate the dictionary.

The results have shown, that it was possible to find such a dictionary between vertex operators in the bulk and their dual scalar operator on the boundary CFT. First, in the free theory the dictionary of vertex operators corresponds in the same way as for bulk fields, which we have shown in the derivation of the boundary propagator. Second, in interaction theory, even after several renormalizations, the corresponding dual operators seem to follow certain constructions such as the anomalous dimension or the appearance of a logarithmic multiplet, giving rise to the assumption of a logarithmic CFT. The found dictionary, which was constructed out of the two-point correlation function, was also able to prescribe the three-point correlation function, which completed the analysis. Similar results have been found in [10] and [9].
One can always stress to formulate higher point correlation functions, in order to fully derivate the theory. But such calculations are marked with time consuming problems that are beyond the scope of this thesis. Yet the dictionary might be profitable in the work of Renormalization Group Flows like in [30] or [40], which analyze the change of an effective description of the physical system, i.e. explaining how the long- and short-range physics are connected. The given heat kernel regularization from the thesis, provides a smooth, ultraviolet regulated propagator. Polchinski's equation for Renormalization Group flows

together with the boundary condition of the effective action provides the necessary tools to analyze perturbative renormalizability or proves the existence and properties of the well-known operator product expansion and so on.

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Appendix A

Conformal Integration

Consider the conformal integral

$$I_\Delta(p_1,p_2) = \int_{\mathrm{PAdS}^2} \mathrm{d}^2\vec{x}\, \sqrt{g} G_{\partial\Delta_1}(\vec{x},p_1) G_{\partial\Delta_2}(\vec{x},p_2), \tag{A.1}$$

where the integration is over the entire Poincaré fundamental domain with the bulk-to-boundary propagator reading

$$G_{\partial\Delta}(\vec{x},p) = \frac{\Gamma(\Delta)}{2\pi^{\frac{1}{2}}\Gamma\left(\Delta+\frac{1}{2}\right)} \left[\frac{z^2+(x-p)^2}{Lz}\right]^{-\Delta}. \tag{A.2}$$

The first step is to use the very useful Schwinger parametrization

$$\Gamma(\Delta)A^{-\Delta} = \int_0^\infty s^{\Delta-1} e^{-As}\,\mathrm{d}s, \qquad [\text{if } \mathrm{Re}(\Delta), \mathrm{Re}(A) > 0], \tag{A.3}$$

which we apply upon the bulk-to-boundary propagator to obtain

$$G_{\partial\Delta}(\vec{x},p) = \frac{1}{2\sqrt{\pi}\Gamma\left(\Delta+\frac{1}{2}\right)} \left(\frac{L_{\mathrm{AdS}}}{z}\right)^{\Delta} \int_0^\infty \mathrm{d}s\, s^{\Delta-1} e^{-s\left[z^2+(p-x)^2\right]}. \tag{A.4}$$

Implement this expression into (A.1) and abusing the fact, that the integration allows us to shift the order of the integrals, where we put the x integration up first and rearrange all x dependent functions under the integral, we are able to evaluate the identified Gaussian Integral

$$I_{\mathrm{Gauss}} = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-s(p_1-x)^2-t(p_2-x)^2} = \sqrt{\frac{\pi}{s+t}} e^{\frac{st(p_1-p_2)^2}{s+t}}, \tag{A.5}$$

with $s+t>0$ and $sp_1+tp_2 \in \mathbb{C}$. Plugging the results back into the integrand of the conformal integral and rescaling the parameter $s \to st$ gives

$$\begin{aligned} I_\Delta(p_1,p_2) = \frac{L_{\mathrm{AdS}}^{2+2\Delta}}{4\sqrt{\pi}\Gamma\left(\Delta+\frac{1}{2}\right)^2} &\int_0^\infty \mathrm{d}z\, z^{2(\Delta-1)} \int_0^\infty \mathrm{d}s\, s^{\Delta-1}(s+1)^{-\frac{1}{2}} \\ &\times \int_0^\infty \mathrm{d}t\, t^{2\Delta-\frac{3}{2}} e^{-t\left[z^2(s+1)+\frac{s(p_1-p_2)^2}{s+1}\right]}. \end{aligned} \tag{A.6}$$

We identify the integration of the parameter t as the Schwinger parametrization (A.3) itself

$$\int_0^\infty \mathrm{d}t\, t^{2\Delta-\frac{3}{2}} e^{-t\left[z^2(s+1)+\frac{s(p_1-p_2)^2}{s+1}\right]} = \Gamma\left(2\Delta-\frac{1}{2}\right)\left[z^2\left(s+1\right)+\frac{s\left(p_1-p_2\right)^2}{s+1}\right]^{\frac{1}{2}-2\Delta}, \tag{A.7}$$

with assumption $[\mathrm{Re}(\Delta) > \frac{1}{4}]$ for convergence. The remaining integrals can be entangled in the s and z integration by using the transformation

$$(A+B)^{-u} = \int_J \frac{\mathrm{d}v}{2\pi i}\frac{\Gamma\left(u-v\right)\Gamma\left(v\right)}{\Gamma\left(u\right)} A^{-v}B^{v-u}, \qquad [\text{if } A, B, \mathrm{Re}(u) > 0], \tag{A.8}$$

with integration interval $J = \{v \in \mathbb{R} \,|\, 0 < v < u\}$ [30]. Hence we set

$$A = \frac{s\left(p_1-p_2\right)^2}{s+1},\; B = z^2\left(s+1\right),\; u = 2\Delta - \frac{1}{2}, \tag{A.9}$$

to evaluate the s integration first

$$\int_0^\infty \mathrm{d}s\, s^{\Delta-v-1}\left(s+1\right)^{2v-2\Delta} = \frac{\Gamma\left(\Delta-v\right)^2}{\Gamma\left(2\Delta-2v\right)}, \tag{A.10}$$

and second the z integration

$$\int_0^\infty \mathrm{d}z\, z^{2v-2\Delta-1} = \lim_{z\to 0}\frac{z^{2v-2\Delta}}{2\Delta-2v}. \tag{A.11}$$

This result suffers heavily from IR divergences, which is why we need to introduce a cut-off in radial direction at $z = \epsilon/M$ so that we define the integral by its finite part

$$\begin{aligned} I_\Delta(p_1,p_2) = \frac{L_{\mathrm{AdS}}^{2\Delta+2}}{4\sqrt{\pi}\Gamma\left(\Delta+\frac{1}{2}\right)^2} \underset{\epsilon\to 0}{\mathcal{P}f}\int_J \frac{\mathrm{d}v}{2\pi i}\left(\frac{\epsilon}{M}\right)^{2v-2\Delta}\left(p_1-p_2\right)^{-2v} \\ \times\frac{\Gamma\left(2\Delta-v-\frac{1}{2}\right)\Gamma\left(v\right)\Gamma\left(\Delta-v\right)^2}{\Gamma\left(2\Delta-2v+1\right)}, \end{aligned} \tag{A.12}$$

where integration interval reads now

$$J = \left\{v \in \mathbb{R} \,|\, 0 < v < \min\left\{\mathrm{Re}(\Delta), 2\mathrm{Re}(\Delta)-\frac{1}{2}\right\}\right\}. \tag{A.13}$$

The finite part of this complex integral can be calculated by the residue theorem.

Definition A.0.1. *Let $f(z)$ be a function which has poles z_j, $j = 1, \ldots, N$ inside a simple closed contour C and no other singularities in this region. Then,*

$$\oint_C \mathrm{d}z\, f(z) = 2\pi i\sum_{j=1}^N \mathrm{Res}[f(z); z_j], \tag{A.14}$$

where the residues of the poles z_0 of order k are computed by

$$\mathrm{Res}[f(z); z_0] = \lim_{z\to z_0}\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\left[\left(z-z_0\right)^k f(z)\right]. \tag{A.15}$$

We find a double pole at $v = \Delta$ inside a simple closed contour C for the integrand

$$f(v) = \left(\frac{\epsilon}{M}\right)^{2v-2\Delta} (p_1 - p_2)^{-2v} (\Delta - v)^{-2} \\ \times \frac{\Gamma\left(2\Delta - \frac{1}{2} - v\right) \Gamma(v) \Gamma(\Delta + 1 - v)^2}{\Gamma(2\Delta + 1 - 2v)}, \tag{A.16}$$

where no other singularities are contributing, therefore

$$\oint_C \frac{\mathrm{d}v}{2\pi i} f(v) = -\mathrm{Res}\left[f(v); \Delta\right]. \tag{A.17}$$

We identify the pole at Δ to be of order two, where we can use the formula in Def. A.0.1 to compute the residue explicitly

$$\begin{aligned} \mathrm{Res}\left[f(v); \Delta\right] &= \lim_{v \to \Delta} \frac{\mathrm{d}}{\mathrm{d}v} \left[(v - \Delta)^2 f(v)\right] \\ &= -\frac{\Gamma(\Delta)\Gamma\left(\Delta + \frac{1}{2}\right)}{(p_1 - p_2)^{2\Delta}} \left[2 \ln \frac{(p_1 - p_2)M}{\epsilon} - \gamma - \psi_0(\Delta) + \psi_0\left(\Delta - \frac{1}{2}\right)\right]. \end{aligned} \tag{A.18}$$

This concludes the calculation of the conformal integral, which was also derived in [30]

$$I_\Delta(p_1, p_2) = \frac{L_{\mathrm{AdS}}^{2\Delta+2}}{2\Delta - 1} \frac{c_\Delta}{|p_1 - p_2|^{2\Delta}} \left[2 \ln \frac{|p_1 - p_2|\, M}{\epsilon} - \gamma - \psi_0(\Delta) + \psi_0\left(\Delta - \frac{1}{2}\right)\right]. \tag{A.19}$$

Declaration of Authorship

I hereby certify that this thesis has been composed by me and is based in my own work, unless stated otherwise. No other person's work has been used without due acknowledgement in this thesis. All references and verbatim extracts have been quoted, and all sources of information have been acknowledged. This work has not been submitted elsewhere in any other form for the fulfillment of any other degree or qualification.

Leipzig, den

Erik Löffelholz